丘成桐数学科学中心
YAU MATHEMATICAL SCIENCES CENTER

Random Walks and Homogenization Theory

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Table of Contents

Outline

Random Walk on Lattice

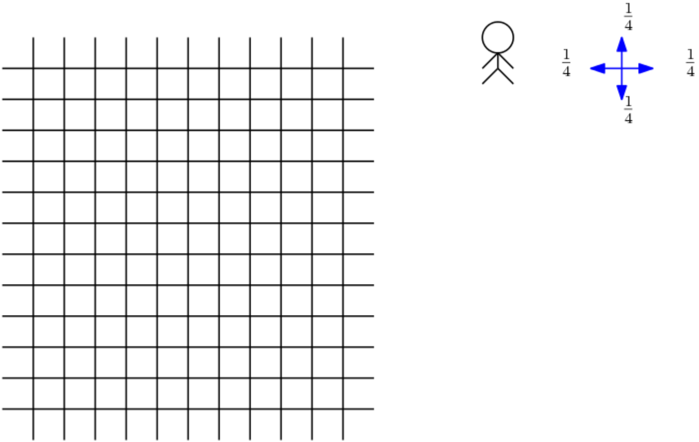


Figure: An illustration for the simple random walk on $\mathbb{Z}^2$.

Random Walk on Lattice

- Consider $\{X_i\}_{i \geqslant 1}$ i.i.d. random variables, and let $S_n := \sum_{i=1}^n X_i$.
- Q: Recurrent or transient ?
- Q: Long time behavior ?

CLT

Theorem (Central Limit Theorem)

For $\{X_i\}_{i\geqslant 1}$ *i.i.d. random variables with* $\mathbb{E}[X_1]=0$ *and* $\mathrm{Var}[X_1]=\sigma^2$*, let* $S_n := \sum_{i=1}^n X_i$*, then*

$$\frac{1}{\sqrt{n}} S_n \Rightarrow \mathcal{N}(0,\sigma^2).$$

Figure: Pierre-Simon Laplace and Johann Carl Friedrich Gauss.

Local CLT

Theorem (Local Central Limit Theorem)

In the setting X centered with finite variance, if X takes value on integer, and aperiodic, then we have local CLT that

$$\lim_{n\to\infty} n^{\frac{d}{2}} \sup_{x\in\mathbb{R}^d} \left| \mathbb{P}[S_n = \lfloor x \rfloor] - \frac{1}{(2\pi n\sigma^2)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{2n\sigma^2}\right) \right| = 0.$$

Q: Convergence rate ?

Random Walk and Brownian Motion

From the view point of process, $(S_n)_{n\geqslant 1}$ can be seen as the trajectory of random walk.

Theorem (Invariance Principle - Donsker's Theorem)

The scaling limit of random walk is Brownian motion.

$$\left(\frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}\right)_{t\geqslant 0} \Rightarrow (\sigma B_t)_{t\geqslant 0}.$$

Q: What is Brownian motion ? Q: Convergence in which topology ?

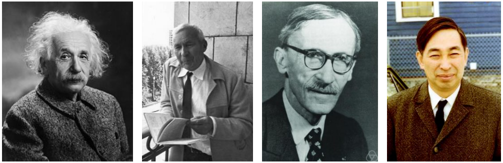
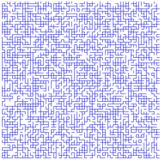
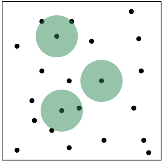
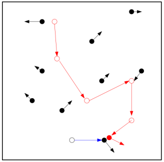
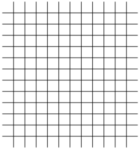
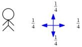


Figure: Albert Einstein, Andrei Kolmogorov, Paul Lévy and Kiyoshi Itô.

Universality of Brownian Motion

- Brownian universality: the limit random variable has the law of Brownian motion despite of the exact law of microscopic behaviors.
- Object: Go beyond the sum of independent random variables. Do these results (CLT, local CLT, invariance principle) also hold for other models (random walk in random environments, particles with interactions, hard-sphere model with collisions, etc)?

Outline

RWRE

- Random walk in random environment: an active topic with rich properties.
  - Random conductance model, random walk with reinforcement, dynamic environment random walk, etc.
  - Random walk on random graphs: Erdös-Renyi graph, percolation (short range, long range, continuum...), trees (regular trees, Galton-Watson trees, uniform spanning trees...), planar maps (uniform, Boltzmann, decorated ...).
- Quenched v.s. Annealed: behavior for almost every environment or that for averaged environment.

RWRE 1d

- Sample i.i.d. random variables $\{\omega_x\}_{x\in\mathbb{Z}}$ and $\omega \in [c, 1-c]$ with $c \in (0,1)$.
- Define a Markov chain starting from 0

$$\mathbb{P}^{\omega}[X_{n+1} = x+1 | X_n = x] = \omega_x,$$
$$\mathbb{P}^{\omega}[X_{n+1} = x-1 | X_n = x] = 1-\omega_x.$$

Theorem (RWRE 1d)

Let $\rho_x := \frac{1-\omega_x}{\omega_x}$*, then*

$$\begin{aligned}
\mathbb{E}[\log \rho_0] < 0 &\Longrightarrow \lim_{n\to\infty} X_n = +\infty, \quad \mathbb{P}\text{-a.s.},\\
\mathbb{E}[\log \rho_0] > 0 &\Longrightarrow \lim_{n\to\infty} X_n = -\infty, \quad \mathbb{P}\text{-a.s.},\\
\mathbb{E}[\log \rho_0] = 0 &\Longrightarrow \limsup_{n\to\infty} X_n = +\infty,\\
&\qquad \liminf_{n\to\infty} X_n = -\infty \quad \mathbb{P}\text{-a.s.}.
\end{aligned}$$

RWRE 1d

Theorem (1D RWRE)

$$\lim_{n\to\infty} \frac{X_n}{n} = \begin{cases} \frac{1-\mathbb{E}[\rho_0]}{1+\mathbb{E}[\rho_0]} & \mathbb{E}[\rho_0] < 1, \\ 0 & \mathbb{E}[\rho_0] \geqslant 1 \textit{ and } \mathbb{E}[\rho_0^{-1}] \geqslant 1, \\ -\frac{1-\mathbb{E}[\rho_0^{-1}]}{1+\mathbb{E}[\rho_0^{-1}]} & \mathbb{E}[\rho_0^{-1}] < 1. \end{cases}$$

Random Conductance Model

- Sample i.i.d. random conductance $\{\mathbf{a}(e)\}_{e \in E_d}$.
- Let $(Y_t)_{t \geqslant 0}$ be a continuous-time Markov jump process starting from y, with an associated generator either
  - variable speed random walk VSRW

$$L_V^{\mathbf{a}} u(x) := \sum_{z \sim x} \mathbf{a}(\{x, z\}) \left(u(z) - u(x)\right);$$

  - constant speed random walk CSRW

$$L_C^{\mathbf{a}} u(x) := \sum_{z \sim x} \frac{\mathbf{a}(\{x, z\})}{\pi(x)} \left(u(z) - u(x)\right),$$

  with $\pi(x) := \sum_{z \sim x} \mathbf{a}(\{x, z\})$.

IP for Random Conductance Model

Theorem (Invariance Principle)

When $0 < c \leqslant \mathbf{a} \leqslant C < \infty$, the scaling limit of VSRW or CSRW is Brownian motion.

$$\left(\frac{1}{\sqrt{n}} Y_{nt}\right)_{t \geqslant 0} \Rightarrow (\bar{\sigma} B_t)_{t \geqslant 0}.$$

Q: Why we need the bound of $\mathbf{a}$ and what happens if this condition fails ?

Corrector Method

- Identification of the limit: the corrector ϕ_{e_i} such that $L_V^{\mathbf{a}}(e_i + \nabla \phi_{e_i}) = 0$. Then we have

$$M_t = \left(Y_t \cdot e_1 + \phi_{e_1}(Y_t), \cdots, Y_t \cdot e_d + \phi_{e_d}(Y_t)\right),$$

is a martingale and the martingale convergence theorem applies

$$\left(\frac{1}{\sqrt{n}} M_{nt}\right)_{t \geqslant 0} \overset{n \to \infty}{\Longrightarrow} (\bar{\sigma} B_t)_{t \geqslant 0}.$$

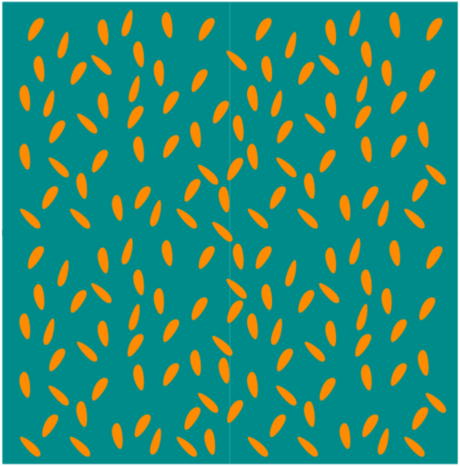
- Corrector is sublinear: $\limsup_{x \to \infty} \frac{\phi_{e_i}(x)}{|x|} = 0$, $|Y_{nt}| \simeq \sqrt{nt}$ implies

$$\frac{1}{\sqrt{n}} \phi_{e_i}(Y_{nt}) \overset{n \to \infty}{\longrightarrow} 0.$$

- Q: Why the martingale part will converge to Brownian motion ?
- Q: Why the corrector is sublinear ?

Outline

Heterogeneous Medium



Homogenization Theory

- Elliptic Dirichlet problem with random, symmetric, $\mathbb{Z}^d$-stationary and ergodic coefficient in a large domain

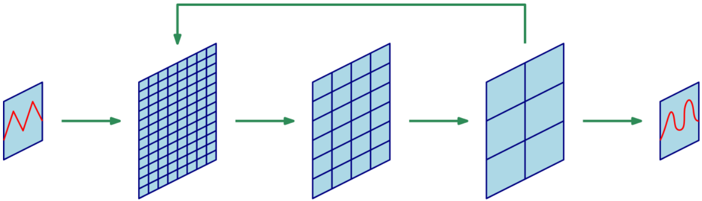
$$\begin{cases} -\nabla \cdot (\mathbf{a} \nabla u) = f & \text{in } Q_r, \\ u = g & \text{on } \partial Q_r. \end{cases}$$

- For very large r, the solution can be approximated by the homogenized solution $\bar{u}$ for

$$\begin{cases} -\nabla \cdot (\bar{\mathbf{a}} \nabla \bar{u}) = f & \text{in } Q_r, \\ \bar{u} = g & \text{on } \partial Q_r, \end{cases}$$

where $\bar{\mathbf{a}} \in \mathbb{R}^{d \times d}$ is the (deterministic) effective coefficient.

Viewpoint from PDE



Probabilistic Interpretation

- $L_V^{\mathbf{a}}$ can also be considered as a discrete divergence form $\nabla \cdot \mathbf{a} \nabla$.
- Find an algorithm to solve the elliptic Dirichlet problem quickly for **big** r,

$$\begin{cases} -\nabla \cdot \mathbf{a} \nabla u = 0 & \text{in } \mathrm{int}(Q_r), \\ u = g & \text{on } \partial Q_r. \end{cases}$$

- Probabilistic representation is $\mathbb{E}^{\mathbf{a}}[g(Y_\tau)]$ for the hitting time τ of the boundary, which should be very close to that of "$\mathbb{E}[g(\bar{\sigma} B_\tau)]$", the solution of the Dirichlet problem

$$\begin{cases} -\nabla \cdot \bar{\mathbf{a}} \nabla \bar{u} = 0 & \text{in } Q_r, \\ \bar{u} = g & \text{on } \partial Q_r, \end{cases}$$

with $\bar{\mathbf{a}} = \frac{1}{2}\bar{\sigma}^2$.

Qualitative Homogenization

Theorem (Homogenization)

For periodic coefficient or stationary ergodic coefficient, we have

$$u \simeq \bar{u} \text{ in } L^2,$$
$$Gradient : \nabla u \simeq \nabla \bar{u} \text{ in } H^{-1},$$
$$Flux : \mathbf{a}\nabla u \simeq \bar{\mathbf{a}}\nabla \bar{u} \text{ in } H^{-1}.$$

Q: What is the convergence rate ?

Quantitative Stochastic Homogenization

- Generally, $\bar{\mathbf{a}} \neq \mathbb{E}[\mathbf{a}]$.
- $\bar{\mathbf{a}}$ is the limit of the averaged Dirichlet energy

$$\nu(Q_r, p) := \inf_{\phi \in H_0^1(Q_r)} \frac{1}{|Q_r|} \int_{Q_r} \frac{1}{2}(p + \nabla\phi) \cdot \mathbf{a}(p + \nabla\phi)$$

$$\nu(Q_r, p) = \frac{1}{2} p \cdot \bar{\mathbf{a}}(Q_r) p,$$

$$\bar{\mathbf{a}} = \lim_{r \to \infty} \bar{\mathbf{a}}(Q_r).$$

Theorem (Stochastic Homogenization)

There exists $\alpha \in (0, \infty)$, such that $|\bar{\mathbf{a}} - \bar{\mathbf{a}}(Q_r)| \simeq r^{-\alpha}$.

- Q: How can we measure this stochastic error ?
- Q: What can we deduce from this convergence ?

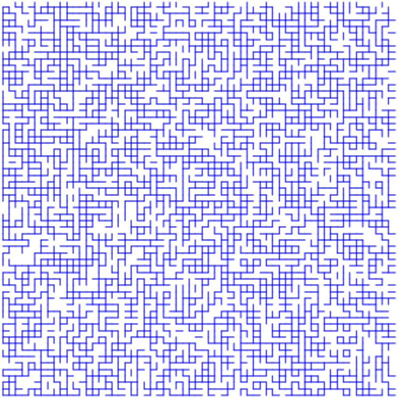
History

- Qualitative homogenization: 1970-2000.
- Quantitative homogenization: 2000-present.

Figure: Some researchers who contribute to homogenization theory: Alain Bensoussan, Jacques-Louis Lions, George Papanicolaou, Ennio De Giorgi, François Murat, Luc Tartar, Thomas Spencer, S. R. Srinivasa Varadhan, Tatsien Li, Grégoire Allaire, Marco Avellaneda, Carlos Kenig, Fanghua Lin, Zhongwei Shen, Felix Otto, Antoine Gloria, Stefan Neukamm, Scott Armstrong, Charles Smart, Jean-Christophe Mourrat, Tuomo Kuusi.

Outline

Random walk in the labyrinth



Question: What happens for the random walk in the labyrinth ?

Definition

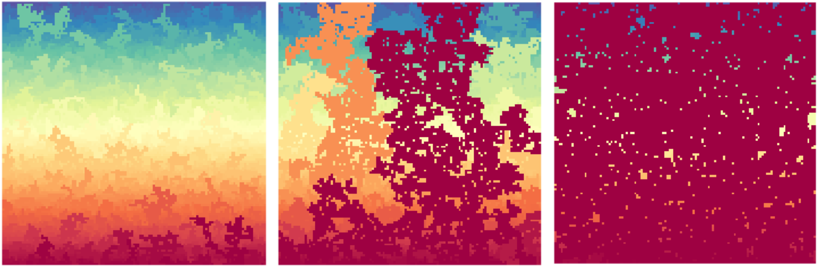
Definition (Bernoulli percolation on $\mathbb{Z}^d$)

We denote by $(\mathbb{Z}^d, E_d)$ the d-dimension lattice graph. A Bernoulli percolation configuration $\{\mathbf{a}(e)\}_{e \in E_d}$ is an element of $\{0,1\}^{E_d}$, and its law is given by

$$\{\mathbf{a}(e)\}_{e \in E_d} \text{ i.i.d. }, \mathbb{P}[\mathbf{a}(e) = 1] = 1 - \mathbb{P}[\mathbf{a}(e) = 0] = \mathfrak{p}.$$

We say that the edge e is open if $\mathbf{a}(e) = 1$ and the edge e is closed if $\mathbf{a}(e) = 0$. A connected component given by $\mathbf{a}$ will be called cluster.

Example of percolation ($\mathfrak{p} = 0.4, 0.5, 0.6$)



Phase transition

- $\theta(\mathfrak{p}) := \mathbb{P}[0$ belongs to an infinite cluster $\mathscr{C}_\infty]$.
- It is easy to show that $\theta(\mathfrak{p})$ is monotone.
- $\mathfrak{p}_c := \inf\{\mathfrak{p} \in [0,1] : \theta(\mathfrak{p}) > 0\}$.

Theorem (Broadbent, Hammersley 57)

For $d \geqslant 2$, we have $0 < \mathfrak{p}_c < 1$.

- We call the regime $0 \leqslant \mathfrak{p} < \mathfrak{p}_c$ subcritical, $\mathfrak{p} = \mathfrak{p}_c$ critical and $\mathfrak{p}_c < \mathfrak{p} \leqslant 1$ supercritical.
- Furthermore, by Kolmogorov 0-1 law, in subcritical case a.s. there is no infinite cluster. In supercritical case a.s. there exists a unique infinite cluster $\mathscr{C}_\infty$.
- Critical case: we conjecture $\theta(\mathfrak{p}_c) = 0$, but it is open for $3 \leqslant d \leqslant 10$.

Infinite cluster $\mathscr{C}_\infty$ in supercritical percolation

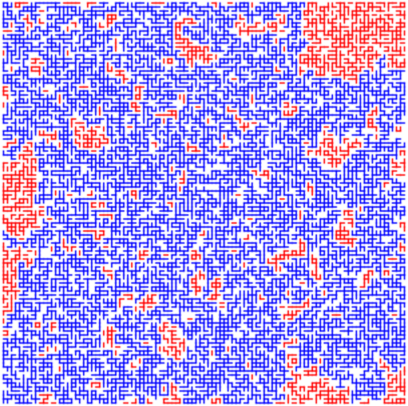


Figure: The cluster in blue is the maximal cluster in the cube.

IP on Percolation

Theorem (Invariance Principle)

The invariance principle also holds for VSRW and CSRW on the infinite cluster of supercritical percolation.

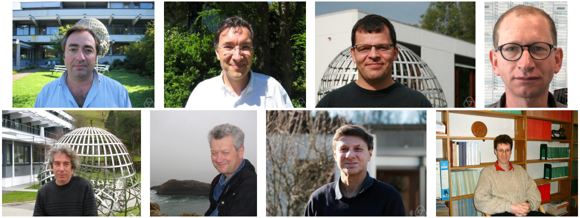


Figure: Vladas Sidoravicius, Alain-Sol Sznitman, Marek Biskup, Noam Berger, Pierre Mathieu, Andrey Piatnitski, Martin Barlow, Ben Hambly.

Random Walk on Percolation

- We focus on the case supercritical percolation.
- (X_t) is a continuous-time Markov jump process starting from $y \in \mathscr{C}_\infty$, with an associated generator

$$\nabla \cdot \mathbf{a}\nabla u(x) := \sum_{z \sim x} \mathbf{a}(\{x,z\})\left(u(z) - u(x)\right).$$

- The quenched semigroup is defined as

$$p\left(t,x,y\right) = p^{\mathbf{a}}\left(t,x,y\right) := \mathbb{P}^{\mathbf{a}}_y\left(X_t = x\right),$$

which also solves the equation on $\mathscr{C}_\infty$ that

$$\begin{cases} \partial_t p\left(t,\cdot,y\right) - \nabla \cdot \mathbf{a}\nabla p\left(t,\cdot,y\right) = 0 & , \\ p\left(0,\cdot,y\right) = \delta_y(\cdot) & . \end{cases}$$

Quantitative Local CLT on Percolation

Theorem (Dario, Gu, AOP 2021)

For each exponent $\delta > 0$, there exist a positive constant $C(d, \mathfrak{p}, \delta) < \infty$ and an exponent $s(d, \mathfrak{p}, \delta) > 0$, such that for every $y \in \mathbb{Z}^d$, there exists a non-negative random time $\mathcal{T}_{\mathrm{par},\delta}(y)$ satisfying the stochastic integrability estimate

$$\forall T \geqslant 0, \ \mathbb{P}\left(\mathcal{T}_{\mathrm{par},\delta}(y) \geqslant T\right) \leqslant C \exp\left(-\frac{T^s}{C}\right),$$

such that, on the event $\{y \in \mathscr{C}_\infty\}$, for every $x \in \mathscr{C}_\infty$ and every $t \geqslant \max\left(\mathcal{T}_{\mathrm{par},\delta}(y), |x-y|\right)$,

$$\left|p(t,x,y) - \theta(\mathfrak{p})^{-1}\bar{p}(t,x-y)\right| \leqslant C t^{-\frac{d}{2}-\left(\frac{1}{2}-\delta\right)} \exp\left(-\frac{|x-y|^2}{Ct}\right).$$

Remark: $\theta(\mathfrak{p}) = \mathbb{P}[0 \in \mathscr{C}_\infty]$ is the factor of the density normalization. $(\bar{p}(t, \cdot - y))_{t \geqslant 0}$ is the semigroup of the limit Brownian motion $(\bar{\sigma} B_t)_{t \geqslant 0}$.

$t^{\frac{d}{2}} p(t, \cdot, 0)$ vs. $t^{\frac{d}{2}} \left| p(t, \cdot, 0) - \theta(\mathfrak{p})^{-1} \bar{p}(t, \cdot) \right|$

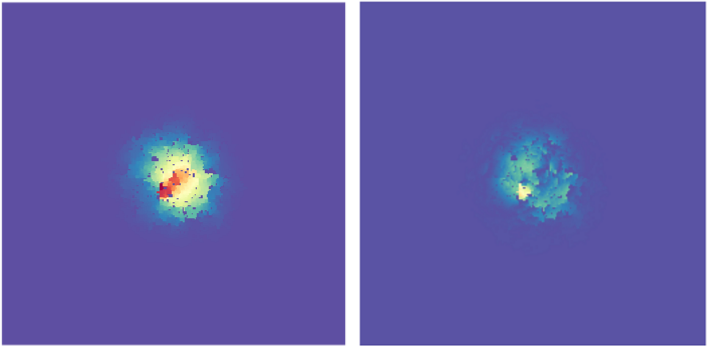


Figure: $t = 500$.

$t^{\frac{d}{2}} p(t,\cdot,0)$ vs. $t^{\frac{d}{2}} \left| p(t,\cdot,0) - \theta(\mathfrak{p})^{-1}\bar{p}(t,\cdot) \right|$

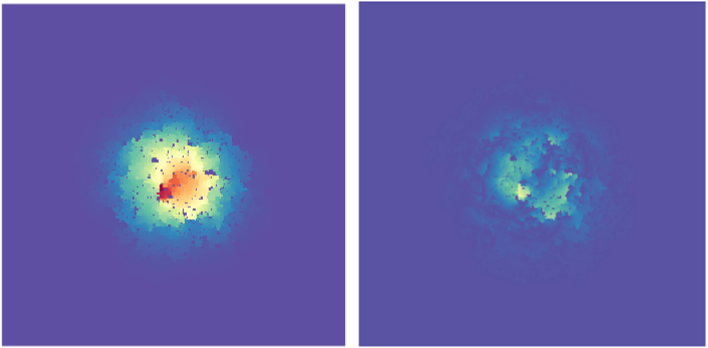


Figure: $t = 1000$.

$t^{\frac{d}{2}} p(t, \cdot, 0)$ vs. $t^{\frac{d}{2}} \left| p(t, \cdot, 0) - \theta(\mathfrak{p})^{-1} \bar{p}(t, \cdot) \right|$

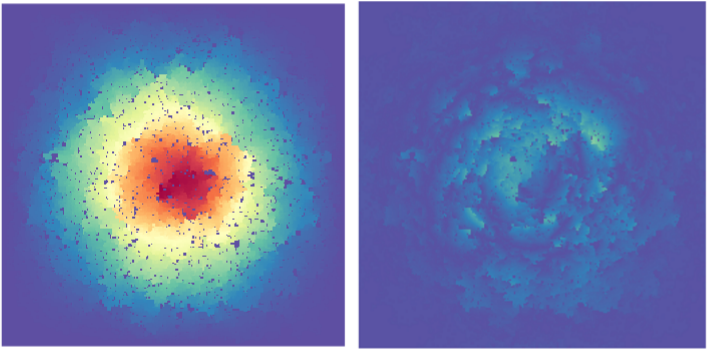


Figure: $t = 4000$.

Two-Scale Expansion

- Two-scale expansion

$$w(t,x,y) = \underbrace{\bar{p}(t,x,y)}_{\text{0th order}} + \underbrace{\sum_{k=1}^{d} \partial_k \bar{p}(t,x,y)\phi_{e_k}(x)}_{\text{1st order}},$$

where $\{\phi_{e_k}\}_{1 \leqslant k \leqslant d}$ is the collection of corrector solving

$$\nabla \cdot \mathbf{a}(e_k + \nabla \phi_{e_k}) = 0 \qquad \text{on } \mathscr{C}_\infty.$$

- The corrector is sublinear $\phi_{e_k}(x) = o(|x|)$, and has nice cancellation property for its centered flux $\mathbf{g}_{e_k} := \mathbf{a}(e_k + \nabla \phi_{e_k}) - \frac{\bar{\sigma}^2}{2} e_k$.
- w is close both to $\bar{p}$ and to p:

$$w - \bar{p} = \sum_{k=1}^{d} \partial_k \bar{p}(t,x,y)\phi_{e_k} \simeq |\nabla \bar{p}|,$$

$$(\partial_t - \nabla \cdot \mathbf{a}\nabla)(w - p) \simeq \mathbf{g}_{e_k}.$$

Two-Scale Expansion

$(w - p)$ is small in the weak sense.

Step 1 : Establishing the equation for w. We claim that the function w satisfies the equation

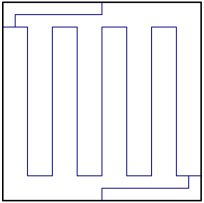
$$\begin{cases} \partial_t w(\cdot,\cdot,\tau,y) - \nabla \cdot \mathbf{a} \nabla w(\cdot,\cdot,\tau,y) = f(\cdot,\cdot,y) + \mathcal{D}^* \cdot F(\cdot,\cdot,y) + \xi(\cdot,\cdot,y) & \text{in } (\tau,\infty) \times \mathscr{C}_\infty, \\ w(\tau,\cdot,y) = 0 & \text{in } \mathscr{C}_\infty, \end{cases} \tag{4.25}$$

where the three functions $f : (0,\infty) \times \mathscr{C}_\infty \times \mathscr{C}_\infty \to \mathbb{R}$, $F : (0,\infty) \times \mathscr{C}_\infty \times \mathscr{C}_\infty \to \mathbb{R}^d$ and $\xi : (0,\infty) \times \mathscr{C}_\infty \times \mathscr{C}_\infty \to \mathbb{R}$ are defined by the formulas, for each $(t,y) \in (0,\infty) \times \mathscr{C}_\infty$,

$$\begin{cases} f(t,\cdot,y) = \frac{1}{2}\bar{\sigma}^2 \left(\Delta \bar{p}(t,\cdot - y) - (-\mathcal{D}^* \cdot \mathcal{D}\bar{p}(t,\cdot - y)) \right) + \sum_{k=1}^{d} \left(\partial_t \mathcal{D}_{e_k} \bar{p}(t,\cdot - y) \right) \chi_{e_k}(\cdot), \\ [F]_i(t,\cdot,y) = \sum_{k=1}^{d} [\mathbf{a}\mathcal{D}\mathcal{D}_{e_k}\bar{p}(t,\cdot - y)]_i T_{e_i}(\chi_{e_k})(\cdot), \qquad \forall i \in \{1,\ldots,d\}, \\ \xi(t,\cdot,y) = \sum_{k=1}^{d} \mathcal{D}^* \mathcal{D}_{e_k} \bar{p}(t,\cdot - y) \cdot \widetilde{\mathbf{g}}^*_{e_k}(\cdot), \end{cases} \tag{4.26}$$

Challenge on Percolation Cluster

- Delmotte (1999) proves the Gaussian bound for Markov chain on graph satisfying the double volume condition and the Poincaré inequality.
- However, the Poincaré inequality is perturbed by the random geometry of the cluster, and the uniform ellipticity is also broken.



- In the work of Barlow (2004), he introduces the idea of good cube in percolation cluster.
- This technique is improved in the work of Armstrong and Dario (2018).

Partition of Good Cube

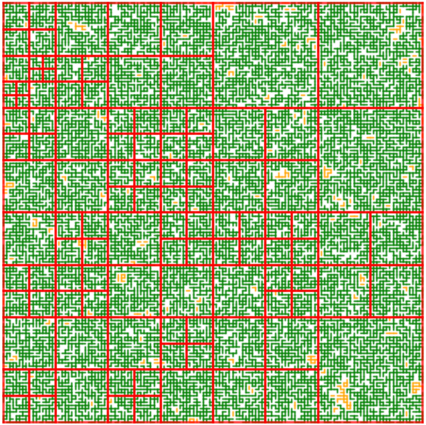


Figure: Decomposition of a big cube into of disjoint small cubes with good properties Armstrong, Dario (2018).

Further Discussions

For Further Reading I

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For Further Reading II

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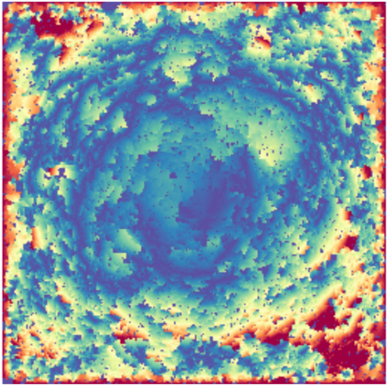

Thank you for your attention.