MATH-UA.0224@NYU

Final Exam

05/13/2020 Time: 120 min

Name: *Univ ID:*

In the following 5 questions, every question counts 5 points and please write down the main steps of the solution. The final score will be truncated by 20 i.e. $\min\{\text{score}, 20\}/20$.

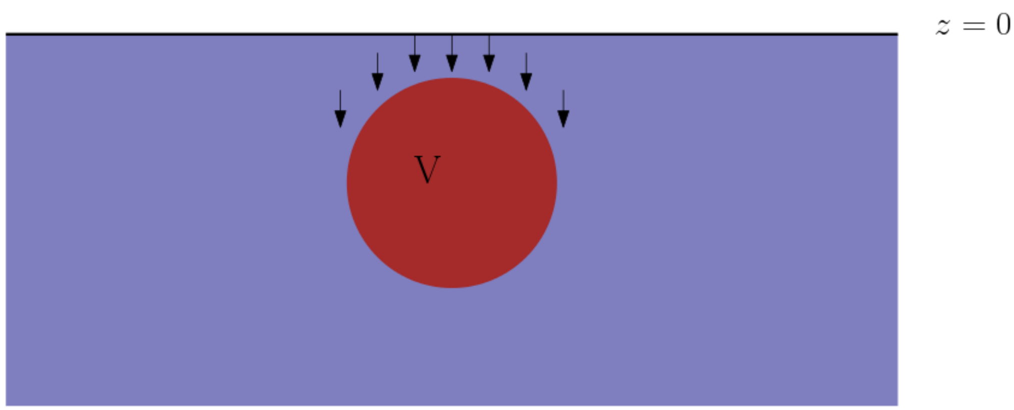
Personal lecture notes are allowed for the exam.

Question 1 (5 pts)**.** *For $f \in C^\infty(\mathbb{R}^3)$ and $F : \mathbb{R}^3 \to \mathbb{R}^3$ a smooth vector field, prove that $\nabla \times \nabla f = 0$ and $\nabla \cdot (\nabla \times F) = 0$.*

Question 2 (5 pts)**.** *Let V be a compact 3-dimensional manifold with smooth boundary in the lower half-space $z < 0$ of $\mathbb{R}^3$. Think of V as a non-compressible object submerged in a fluid of uniform density ρ on its. The buoyant force B on V, due to the fluid, is defined by*

$$B = -\int_{\partial V} F \cdot n,$$

where $F = (0, 0, \rho z)$. Prove the Archimedes law that $B = -\int_V \rho$.



Question 3 (5 pts)**.** *Let $(\alpha, \beta) \in [0, 2\pi]^2$ and $(\alpha, \beta) \mapsto (x, y, z)$ be the parametrization of torus*

$$\begin{aligned} x &= (R + r\cos\alpha)\cos\beta, \\ y &= (R + r\cos\alpha)\sin\beta, \\ z &= r\sin\alpha. \end{aligned}$$

Calculate its area and volume.

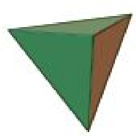
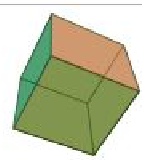
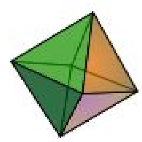
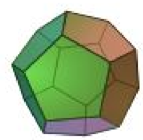
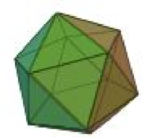
Question 4 (5 pts)**.** *The function of winding number is defined as an integral of* 1*-form* ω *defined on* $\mathbb{R}^2 \backslash \{0\}$*.*

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy. \tag{1}$$

Let γ *be the closed curve* ∂B_1*.*

1. *Calculate* $\int_{\partial B_1} \omega d\gamma$*.*
2. *Can we apply the Green's theorem for the integral above? Why?*

Question 5 (5 pts)**.** *In three-dimensional space, a Platonic solid is a regular, convex polyhedron. It is constructed by congruent (identical in shape and size), regular (all angles equal and all sides equal), polygonal faces with the same number of faces meeting at each vertex. Prove that there are 5 types of Platonic solid.*

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Four faces	Six faces	Eight faces	Twelve faces	Twenty faces
				
(Animation) (3D model)	(Animation) (3D model)	(Animation) (3D model)	(Animation) (3D model)	(Animation) (3D model)