# Lecture 4: Differential Form on $\mathbb{R}^d$

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# Outline for section 1

1 k-form
- Definition
- Structure

2 Operations on k-form
- Addition
- Exterior Product
- Exterior Differential
- Pull-Back

# Object

- Object: A language of multi-linear algebra to unify vector analysis.
- Something we have learned in the previous lecture:
  - Integral along regular curve: integral of 1-form

$$\int_\gamma F\,d\gamma := \int_a^b F(t)\cdot\gamma'(t)\,dt = \sum_{i=1}^{d}\int_a^b F_i(t)\gamma_i'(t)\,dt.$$

  - Green's theorem: transform an integral of 1-form to 2-form

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial D} P\,dx + Q\,dy. \tag{1.1}$$

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# Multi-linear algebra

**Definition**

$\phi : (\mathbb{R}^n)^k \to \mathbb{R}$ is multi-linear form $((\mathbb{R}^n)^k)^*$ if and only if that

$$\phi(v_1, v_2 \cdots \alpha v_i + \beta v_i' \cdots v_k) = \alpha\phi(v_1, v_2 \cdots v_i \cdots v_k) + \beta\phi(v_1, v_2 \cdots v_i' \cdots v_k).$$

- $\mathsf{dim}((\mathbb{R}^n)^*) = n$.
- Easy to prove that $\mathsf{dim}(((\mathbb{R}^n)^k)^*) = nk$.

# k-alternating-form

Definition (k-alternating-form)

$\varphi : (\mathbb{R}^n)^k \to \mathbb{R}$ is a k-alternating-form $\Lambda^k(\mathbb{R}^n)^*$ if it is multi-linear form and once we interchange an argument, we add a factor $(-1)$ i.e.

$$\varphi(v_1, v_2 \cdots v_i, v_{i+1} \cdots v_k) = (-1)\varphi(v_1, v_2 \cdots v_{i+1}, v_i \cdots v_k).$$

Question: Dimension and basis of $\Lambda^k(\mathbb{R}^n)^*$.

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# k-form

**Theorem**

We have that $\mathsf{dim}(\Lambda^k(\mathbb{R}^n)^*) = \binom{n}{k}$, and let $\{dx_{\alpha_1} \wedge dx_{\alpha_2} \cdots \wedge dx_{\alpha_k}\}_{1 \leqslant \alpha_1 < \alpha_2 \cdots < \alpha_k \leqslant n}$ be a basis such that

$$dx_{\alpha_1} \wedge dx_{\alpha_2} \cdots \wedge dx_{\alpha_k}(v_1, v_2, \cdots v_k) := \det\left(([v_i]_{\alpha_j})_{1 \leqslant i,j \leqslant k}\right).$$

# k-form

Proof.

If we admit the structure of the basis, then the dimension is clear, which should be "choosing k different indexes among n".

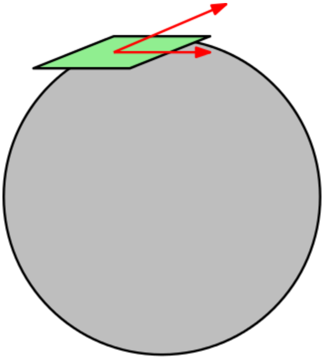
Let $\{e_i\}_{1\leqslant i\leqslant n}$ be the basis in $\mathbb{R}^n$, then it is easy to check that $\{dx_{\alpha_1} \wedge dx_{\alpha_2} \cdots \wedge dx_{\alpha_k}\}_{1\leqslant\alpha_1<\alpha_2\cdots<\alpha_k\leqslant n}$ is linear independent by testing $\{e_i\}_{1\leqslant i\leqslant n}$. Moreover, we have an explicit formula for the expression of the $k-$form:

$$\varphi = \sum_{1\leqslant\alpha_1<\alpha_2\cdots<\alpha_k\leqslant n} \varphi(e_{\alpha_1}, e_{\alpha_2}, \cdots e_{\alpha_k}) dx_{\alpha_1} \wedge dx_{\alpha_2} \cdots \wedge dx_{\alpha_k}.$$

This proves that they are basis of $\Lambda^k(\mathbb{R}^n)^*$. □

- We will make it more clear the sense of $\wedge$.
- For short, one can write down that $\varphi = \sum_I \alpha_I dx_I$, where I denote the multi-index $I = (\alpha_1, \alpha_2 \cdots \alpha_k)$.

# k-form



# k-form

- For a point $p \in U \subset \mathbb{R}^d$, we have $\varphi_p \in \Lambda^k(\mathbb{R}^n_p)^*$ if the vector space is the one associated to p.
- $\varphi = \sum_I \alpha_I dx_I$ is continuous $(C^1)$ if for any I, $\alpha_I$ is continuous $(C^1)$.

# Example of k-form

- $k = 0$, just function in $\mathbb{R}^n$.
- $k = 1, n = 2$, $\Lambda^1(\mathbb{R}^2) : \varphi = \alpha dx + \beta dy$.
- $k = 2, n = 2$, $\Lambda^2(\mathbb{R}^2) : \varphi = \alpha dx \wedge dy$.
- $k = 1, n = 3$, $\Lambda^1(\mathbb{R}^3) : \varphi = \alpha dx + \beta dy + \gamma dz$.
- $k = 2, n = 3$, $\Lambda^2(\mathbb{R}^3) : \varphi = \alpha dx \wedge dy + \beta dy \wedge dz + \gamma dz \wedge dx$.
- $k = 3, n = 3$, $\Lambda^3(\mathbb{R}^3) : \varphi = \alpha dx \wedge dy \wedge dz$.

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# Addition

For two k-forms, $\omega, \varphi \in \Lambda^{k}(\mathbb{R}^{n})^{*}$ that

$$\omega = \sum_{I} \alpha_{I}\, dx_{I}, \qquad \varphi = \sum_{I} \beta_{I}\, dx_{I},$$

we define that

$$\omega + \varphi := \sum_{I} (\alpha_{I} + \beta_{I})\, dx_{I}.$$

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# Exterior Product ($\wedge$)

Let $\omega \in \Lambda^{\mathrm{k}}(\mathbb{R}^{\mathrm{n}})^*$ and $\varphi \in \Lambda^{\mathrm{s}}(\mathbb{R}^{\mathrm{n}})^*$ such that

$$\omega = \sum_{\mathrm{I}} \alpha_{\mathrm{I}} \, \mathrm{dx}_{\mathrm{I}}, \qquad \varphi = \sum_{\mathrm{I}} \beta \, \mathrm{dy}_{\mathrm{J}},$$

then we define the notation $\wedge$

$$\omega \wedge \varphi := \sum_{\mathrm{I},\mathrm{J}} \alpha_{\mathrm{I}} \beta_{\mathrm{J}} \, \mathrm{dx}_{\mathrm{I}} \wedge \mathrm{dy}_{\mathrm{J}}.$$

This gives that $\omega \wedge \varphi$ a $(\mathrm{k}+\mathrm{s})$ form.

# Exterior Product ($\wedge$)

Some properties:

1. $(\omega \wedge \varphi) \wedge \theta = \omega \wedge (\varphi \wedge \theta)$.
2. For $\omega$ a k-form, $\varphi, \theta$ s-forms, then $\omega \wedge (\varphi + \theta) = \omega \wedge \varphi + \omega \wedge \theta$.
3. For $\omega$ a k-form, $\varphi$ s-forms, then $\omega \wedge \varphi = (-1)^{ks} \varphi \wedge \omega$.

**Theorem**

Let $\{\varphi_i\}_{1 \leqslant i \leqslant k}$ be k 1-form, then we have

$$\varphi_1 \wedge \varphi_2 \cdots \wedge \varphi_k(v_1, v_2, \cdots v_k) = \det\left((\varphi_i(v_j))_{1 \leqslant i,j \leqslant k}\right).$$

A simple corollary $dx_i \wedge dx_i = 0$.

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# Exterior Differential

- For a 0-form (function), we have

$$\mathrm{d}f = \sum_{i=1}^{n} \partial_{x_i} f \, \mathrm{d}x_i.$$

- For a k-form $\varphi = \sum_I \alpha_I \, \mathrm{d}x_I$, then we can define $\mathrm{d}\varphi$

$$\begin{aligned} \mathrm{d}\varphi &= \sum_I \mathrm{d}\alpha_I \wedge \mathrm{d}x_I \\ &= \sum_I \Big(\sum_{i=1}^{n} \partial_{x_i}\alpha_I \, \mathrm{d}x_i\Big) \wedge \mathrm{d}x_I \\ &= \sum_{i,I} \partial_i \alpha_I \, \mathrm{d}x_i \wedge \mathrm{d}x_I. \end{aligned}$$

# Exterior Differential

Some properties:

1. $\mathrm{d}(\varphi + \omega) = \mathrm{d}\varphi + \mathrm{d}\omega$.
2. For $\varphi$ a k-form, $\mathrm{d}(\varphi \wedge \omega) = \mathrm{d}\varphi \wedge \omega + (-1)^{\mathrm{k}} \varphi \wedge \mathrm{d}\omega$.
3. $\mathrm{d}(\mathrm{d}\varphi) = 0$.

# Example in Green's Theorem

Green's theorem: transform an integral of 1-form to 2-form

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial D} P\,dx + Q\,dy. \tag{2.1}$$

- 1-form: $\varphi = Pdx + Qdy$.
- $$\begin{aligned} d\varphi &= dP \wedge dx + dQ \wedge dy \\ &= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right) \wedge dy \\ &= \frac{\partial P}{\partial y}dy \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx \wedge dy. \end{aligned}$$
- That is why we write $\int_{\partial D} \varphi = \int_D d\varphi$.

# Interpretation of 1-form

- $\mathrm{d}f = \sum_{i=1}^{n} \partial_{x_i} f \, \mathrm{d}x_i$.
- We add the information of the position

$$\mathrm{d}f_p = \sum_{i=1}^{n} \partial_{x_i} f_p \, \mathrm{d}x_i.$$

- $\forall v \in \mathbb{R}^n, \mathrm{d}f_p : \mathbb{R}^n \to \mathbb{R}$. Nothing but the first order differential (which is a linear map).
- Thus the two notations coincide.

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# Pull-Back

Pull-Back is the idea of $df_p$.

- $f : \mathbb{R}^n \to \mathbb{R}^m$ a $C^1$ differential map.
- $df_p : \mathbb{R}^n \to \mathbb{R}^m$.
- $\varphi \in \Lambda^k(\mathbb{R}^m)^*$.
- "Pull-back" means pull the differential form $\mathbb{R}^m$ to that of $\mathbb{R}^n$.
- $f^*\varphi \in \Lambda^k(\mathbb{R}^n)^*$: for $v_1 \cdots v_k \in \mathbb{R}^n$

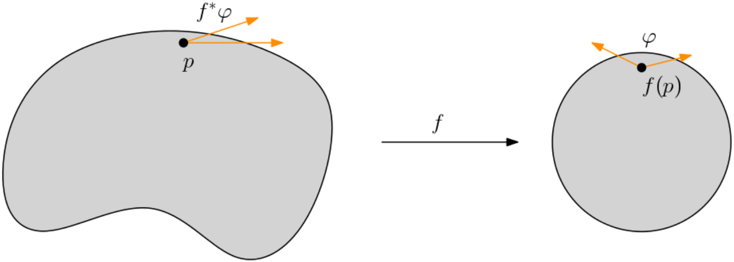
$$\boxed{(f^*\varphi)_p(v_1, v_2, \cdots v_k) := \varphi_{f(p)}(df_p(v_1), df_p(v_2) \cdots df_p(v_k))}. \quad (2.2)$$

# Pull-Back

Some useful properties:

1. $f^*(\varphi + \omega) = f^*\varphi + f^*\omega$.
2. For g a differentiable 0-form, $f^*(g\varphi) = f^*g f^*\varphi$.
3. $f^*(\varphi_1 \wedge \varphi_2 \cdots \wedge \varphi_k) = f^*\varphi_1 \wedge f^*\varphi_2 \cdots \wedge f^*\varphi_k$.
4. $f^*(\varphi \wedge \omega) = f^*\varphi \wedge f^*\omega$.
5. $(f \circ g)^*\varphi = g^*(f^*\varphi)$.
6. $d(f^*\varphi) = f^*(d\varphi)$.

# Pull-Back



# Pull-Back

Prop: For $\{\varphi_i\}_{1\leqslant i\leqslant k}$ 1-form, $f^*(\varphi_1 \wedge \varphi_2 \cdots \wedge \varphi_k) = f^*\varphi_1 \wedge f^*\varphi_2 \cdots \wedge f^*\varphi_k$.

Proof.

$$\begin{aligned}
& f^*(\varphi_1 \wedge \varphi_2 \cdots \wedge \varphi_k)(v_1, v_2 \cdots v_k) \\
=& (\varphi_1 \wedge \varphi_2 \cdots \wedge \varphi_k)(df(v_1), df(v_2) \cdots df(v_k)) \\
=& \det\left(\varphi_i(df(v_j))_{1\leqslant i,j\leqslant k}\right) \\
=& \det\left((f^*\varphi_i(v_j))_{1\leqslant i,j\leqslant k}\right) \\
=& \left(f^*\varphi_1 \wedge f^*\varphi_2 \cdots \wedge f^*\varphi_k\right)\left(v_1, v_2 \cdots v_k\right).
\end{aligned}$$

□

# Pull-Back

Prop: For $\varphi$ a k-form and $\omega$ a s-form, $f^*(\varphi \wedge \omega) = f^*\varphi \wedge f^*\omega$.

Proof.

Let $\varphi = \sum_I \alpha_I \, dx_I, \omega = \sum_J \beta_J \, dy_J$.

$$\begin{aligned} f^*(\varphi \wedge \omega) &= f^*\left(\sum_{I,J} \alpha_I \beta_J \, dx_I \wedge dy_J\right) \\ &= \sum_{I,J} f^*\alpha_I f^*\beta_J \, df_I \wedge df_J \\ &= \left(\sum_I f^*\alpha_I \, df_I\right) \wedge \left(\sum_J f^*\beta_J df_J\right) \\ &= f^*\varphi \wedge f^*\omega. \end{aligned}$$

□