Lecture 6: Application - Maxwell Equations

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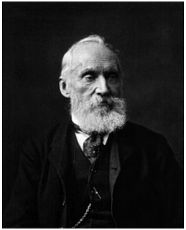
Recap

- Green's theorem.
- Kelvin-Stokes' theorem in $\mathbb{R}^3$, let $F = (P, Q, R)$ and Σ a closed surface, then

$$\int_{\partial\Sigma} F \, d\gamma = \int_{\Sigma} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

- Gauss-Ostrogradsky's theorem: in $\mathbb{R}^3$, let $F = (P, Q, R)$ and S a closed surface and n the normal direction. Then

$$\int_{\partial S} F \cdot n = \int_{S} \nabla \cdot F.$$

Who are they ?

Outline for section 1

Some Notations 1

Some conventions used in physics, let Ω be a bounded open set in $\mathbb{R}^3$, and Σ a bounded surface with boundary $\partial\Sigma$ regular curve.

- $\oiint_{\partial\Omega} \cdot \mathrm{dS}$: integral on the surface = integral of 2-form.
- $\iiint_{\Omega} \mathrm{dV}$: integral triple integral in Ω = integral of 3-form (volume form).
- $\oint_{\partial\Sigma} \mathrm{d}\gamma$: integral along closed regular curve = integral of 1-form.
- $\iint_{\Sigma} \mathrm{dS}$: integral on the surface = integral of 2-form.

Some Notations 2

- $\mathrm{E} : \mathbb{R}^3 \to \mathbb{R}^3$ electric field.
- $\mathrm{B} : \mathbb{R}^3 \to \mathbb{R}^3$ magnetic field.
- ρ : density of electron.
- $\mathrm{J} : \mathbb{R}^3 \to \mathbb{R}^3$ electric flux.
- μ_0, ε_0 constants.

Maxwell Equations - Integral Form

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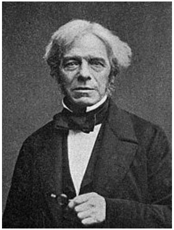
$$\begin{cases} \oiint_{\partial\Omega} \mathrm{E} \cdot \mathrm{dS} = \frac{1}{\varepsilon_0} \iiint_{\Omega} \rho \, \mathrm{dV}, \\ \oiint_{\partial\Omega} \mathrm{B} \cdot \mathrm{dS} = 0, \\ \oint_{\partial\Sigma} \mathrm{E} \, \mathrm{d}\gamma = -\frac{\mathrm{d}}{\mathrm{dt}} \iint_{\Sigma} \mathrm{B} \cdot \mathrm{dS}, \\ \oint_{\partial\Sigma} \mathrm{B} \, \mathrm{d}\gamma = \mu_0 \left(\iint_{\Sigma} \mathrm{J} \cdot \mathrm{dS} + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{dt}} \iint_{\Sigma} \mathrm{E} \cdot \mathrm{dS} \right). \end{cases} \tag{1.1}$$

Maxwell Equations - Differential Form

Maxwell Equations - Differential Form

$$\begin{cases} \nabla \cdot \mathrm{E} = \frac{\rho}{\varepsilon_0}, \\ \nabla \cdot \mathrm{B} = 0, \\ \nabla \times \mathrm{E} = -\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{B}, \\ \nabla \times \mathrm{B} = \mu_0 \left(\mathrm{J} + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{dt}} \mathrm{E} \right). \end{cases} \tag{1.2}$$






Who are they ?

Outline for section 2

Gauss's Law

Gauss's Law

$$\oiint_{\partial\Omega} \mathrm{E} \cdot \mathrm{dS} = \frac{1}{\varepsilon_0} \iiint_{\Omega} \rho \, \mathrm{dV}. \Longleftrightarrow \nabla \cdot \mathrm{E} = \frac{\rho}{\varepsilon_0}. \qquad (2.1)$$

- Gauss's law describes the relationship between a static electric field and the electric charges that cause it: a static electric field points away from positive charges and towards negative charges, and the net outflow of the electric field through any closed surface is proportional to the charge enclosed by the surface.
- By Gauss' formula

$$\oiint_{\partial\Omega} \mathrm{E} \cdot \mathrm{dS} = \iiint_{\Omega} \nabla \cdot \mathrm{E} \, \mathrm{dV},$$

for any regular domain Ω, which gives $\nabla \cdot \mathrm{E} = \frac{\rho}{\varepsilon_0}$ by taking $\Omega \to 0$.

Outline for section 3

Gauss's Law for Magnetism

Gauss's Law for Magnetism

$$\oiint_{\partial\Omega} \mathrm{B} \cdot \mathrm{dS} = 0 \Longleftrightarrow \nabla \cdot \mathrm{B} = 0. \tag{3.1}$$

- Gauss's law for magnetism states that there are no “magnetic charges” (also called magnetic monopoles), analogous to electric charges. Instead, the magnetic field due to materials is generated by a configuration called a dipole, and the net outflow of the magnetic field through any closed surface is zero.
- By Gauss' formula

$$\oiint_{\partial\Omega} \mathrm{B} \cdot \mathrm{dS} = \iiint_{\Omega} \nabla \cdot \mathrm{B} \, \mathrm{dV},$$

for any regular domain Ω, which gives $\nabla \cdot \mathrm{B} = 0$ by taking $\Omega \to 0$.

Outline for section 4

Maxwell-Faraday Equation

Maxwell-Faraday Equation

$$\oint_{\partial\Sigma} \mathrm{E}\,\mathrm{d}\gamma = -\frac{\mathrm{d}}{\mathrm{dt}} \iint_{\Sigma} \mathrm{B} \cdot \mathrm{dS} \Longleftrightarrow \nabla \times \mathrm{E} = -\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{B}. \tag{4.1}$$

- The Maxwell-Faraday version of Faraday's law of induction describes how a time varying magnetic field creates (”induces”) an electric field. In integral form, it states that the work per unit charge required to move a charge around a closed loop equals the rate of change of the magnetic flux through the enclosed surface.
- Using Kelvin-Stokes' theorem, we have

$$\oint_{\partial\Sigma} \mathrm{E}\,\mathrm{d}\gamma = \iint_{\Sigma} \nabla \times \mathrm{E}\,\mathrm{dS}.$$

Then we shrink $\partial\Sigma \to 0$ and prove that $\nabla \times \mathrm{E} = -\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{B}$.

Outline for section 5

Ampère's Circuital Law

Ampère's Circuital Law

$$\oint_{\partial\Sigma} \mathrm{B}\,\mathrm{d}\gamma = \mu_0 \left(\iint_{\Sigma} \mathrm{J} \cdot \mathrm{dS} + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{dt}} \iint_{\Sigma} \mathrm{E} \cdot \mathrm{dS} \right)$$
$$\Longleftrightarrow \nabla \times \mathrm{B} = \mu_0 \left(\mathrm{J} + \varepsilon \frac{\mathrm{d}}{\mathrm{dt}} \mathrm{E} \right). \quad (5.1)$$

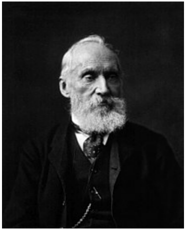
- It states that magnetic fields can be generated in two ways: by electric current (this was the original "Ampère's law") and by changing electric fields (this was "Maxwell's addition", which he called displacement current). In integral form, the magnetic field induced around any closed loop is proportional to the electric current plus displacement current (proportional to the rate of change of electric flux) through the enclosed surface.

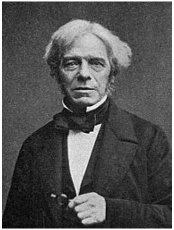
Outline for section 6

In the case $\rho = 0, \mathrm{J} = 0$, we could establish the equation that

$$\begin{cases} \nabla \cdot \mathrm{E} = 0, \\ \nabla \cdot \mathrm{B} = 0, \\ \nabla \times \mathrm{E} = -\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{B}, \\ \nabla \times \mathrm{B} = \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{dt}}\mathrm{E}. \end{cases} \Longrightarrow \begin{cases} \mu_0 \varepsilon_0 \frac{\partial^2}{\partial \mathrm{t}^2}\mathrm{E} = \Delta \mathrm{E}, \\ \mu_0 \varepsilon_0 \frac{\partial^2}{\partial \mathrm{t}^2}\mathrm{B} = \Delta \mathrm{B}. \end{cases}$$




First row: Carl Friedrich Gauss (1777-1855), Mikhail Leo Ostrogradsky(1801-1862), Sir George Stokes (1819-1903), Sir William Thomson - 1st Baron Kelvin(1824-1907).
Second row: Charles-Augustin de Coulomb (1736-1806), André-Marie Ampère (1776-1836), Michael Faraday (1791-1867), James Clerk Maxwell (1831-1879.)