Lecture 7: Application - Introduction to Differential Geometry

Chenlin GU

DMA/ENS, PSL Research University

April 29, 2020

Outline for section 1

Gauss-Bonnet Theorem

Theorem (Gauss-Bonnet Theorem)

Given a Σ a compact smooth surface in $\mathbb{R}^3$, we have

$$\int_{\Sigma} \mathrm{K} \, \mathrm{dS} = 2\pi\chi(\Sigma). \tag{1.1}$$

- Right hand side: Euler characteristic.
- Left hand side: total curvature - integral of Gaussian curvature.

Outline for section 2

Euler Characteristic - Polyhedra

Euler Characteristic

The Euler characteristic χ was classically defined for the surfaces of polyhedra, according to the formula

$$\chi = \mathrm{V} + \mathrm{F} - \mathrm{E}.$$

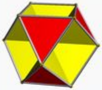
where we have

- V: number of vertex.
- F: number of faces.
- E: number of edges.

Euler Characteristic - Polyhedra Example 1

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

Euler Characteristic - Polyhedra Example 2

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahemihexahedron		6	12	7	**1**
Octahemioctahedron		12	24	12	**0**
Cubohemioctahedron		12	24	10	**−2**
Small stellated dodecahedron		12	30	12	**−6**
Great stellated dodecahedron		20	30	12	**2**

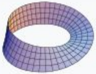
Euler Characteristic - 2D Surface

Theorem (Classification theorem)

The 2D compact orientable smooth surface Σ can be classified by its number of genus g, and the Euler characteristic is

$$\chi(\Sigma) = 2 - 2g. \tag{2.1}$$

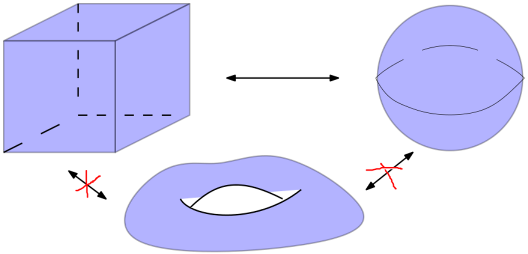
Euler Characteristic - 2D Surface

Sphere		2
Torus (Product of two circles)		0
Double torus		−2
Triple torus		−4
Real projective plane		1
Möbius strip		0

Euler Characteristic - 2D Surface

Theorem

Euler characteristic is invariant up to homeomorphism i.e. bijection and continuous in two directions.



Euler Characteristic - Case for Sphere

- Now $\Sigma = S^2$.
- It suffices to consider the case on the plane.
- Suppose that $\mathbb{R}^2 = \cup_{i=1}^{n} P_i$, with boundary P_0 where $\{P_i\}_{0 \leqslant i \leqslant n}$ are polygons with E_i edges.
-

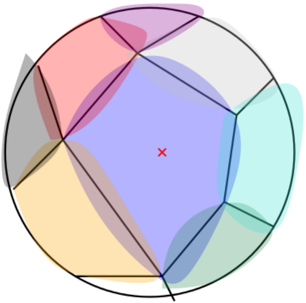
$$2E = \sum_{i=0}^{n} E_i,$$

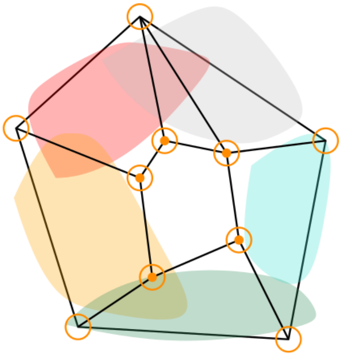
$$\sum_{i=1}^{n}(E_i - 2)\pi = 2\pi(V - E_0) + (E_0 - 2)\pi,$$

$$F = n + 1.$$

$$\Longrightarrow V + F - E = 2.$$

Euler Characteristic - Case for Sphere





Euler Characteristic - Application

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Four faces	Six faces	Eight faces	Twelve faces	Twenty faces
				
(Animation)	(Animation)	(Animation)	(Animation)	(Animation)
(3D model)	(3D model)	(3D model)	(3D model)	(3D model)

Outline for section 3

Outline

Outline

Outline for section 4

Local Gauss-Bonnet theorem

The key idea is the local version of Gauss-Bonnet theorem.

$$\int_{\Sigma} \mathrm{K}\,\mathrm{dS} = (2-\mathrm{n})\pi - \int_{\partial\Sigma} \mathrm{k_g} + \sum_{\mathrm{i}} \theta_{\mathrm{i}}.$$