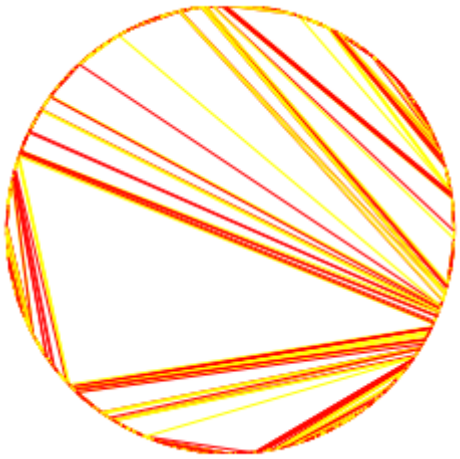
MAP571
Approfondissement en mathématiques
apllquées

Random Recursive Triangulation



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1 Introduction

In this report, we introduce the study of random recursive triangulation on a unit disk. We start from the main model of recursive lamination: In a unit disk $\overline{\mathbb{D}}$, we take a series of i.i.d random variable $\{U_n\}, \{V_n\}$ which take values uniformly on its boundary $\mathbb{S}_1$ and we denote $[U_nV_n]$ the chord in the disk by connecting two endpoints. Then we construct the lamination by recurrence, that is $L_0 = \emptyset$ and $L_{n+1} = L_n \cup [U_{n+1}V_{n+1}]$ if $L_n \cap [U_{n+1}V_{n+1}] \cap \mathbb{D} = \emptyset$, otherwise $L_{n+1} = L_n$ if they intersect. We repeat this process and define the limit object as

$$L_\infty = \overline{\bigcup_{n=1}^{\infty} L_n}$$

, which is a random compact subset of $\overline{\mathbb{D}}$.

It is obvious that in this process, it becomes harder and harder to add a new chord to the disk and it arises naturally a question:

Question 1. How many chord is there in L_n? Does it has asymptotic limit with respect to n?

In numerical simulation, we will see that by normalisation of $\sqrt{n}$, the number converges. We use $N(L_n)$ to represent the number of chord in L_n, then the following theorem is true:

Theorem 1.1.

$$n^{-1/2}N(L_n) \xrightarrow[n\to\infty]{a.s} \sqrt{\pi}$$

The number of chord is just one aspect of this process, it measures how many chords there are in the disk, but it tells little about how hard to add one chord or how dense L_n is. A possible way to measure it is to count the number of chord encountered if we draw a chord connecting 1 and another point x on the circle. This arises a second question:

Question 2. How many chord will it cut in L_n if we connect 1 and another point x? Does it have a distribution when n goes to infinity?

We wonder more in the geometric aspect, although it is a little hard to describe:

Question 3. How can we describe the geometry when n goes to infinity? Obviously, it is a random object, but what is its law?

These three questions give a good example in the research of random geometry, which always try do study the properties of scaling limit and address the connection between the discrete models and continuous model. However, the third question is more difficult to answer since we haven't make it clear and in fact, to describe the geometry, we have to treat it in various views.

Here, we state at first the main theorem by using as less notation as possible. We denote the number of chords that intersect $[xy], x, y \in \mathbb{S}_1$ by $H_n(x, y)$. We note also an important constant in this report

$$\beta^* = \frac{\sqrt{17} - 3}{2}$$

The following theorem answers 2.

Theorem 1.2. *There exists a random process $\mathscr{M}_\infty(x), x \in \mathbb{S}_1$ such that*

$$n^{-\beta^*} H_n(1,x) \xrightarrow[n\to\infty]{\mathbb{P}} \mathscr{M}_\infty(x)$$

We are interested by the process $\mathscr{M}_\infty$ defined above, because we observe that in L_n, $H_n(1,x) = H_n(1,y)$ for an arc without any endpoint of the chord in the middle. This inspires us to conjecture that maybe $\mathscr{M}_\infty$ codes the L_∞. This is the fact and one aspect to study the 3. We use the notation $Arc(x,y)$ to represent the arc which doesn't contain 1 and $Arc^*(x,y)$ the one contains 1. Then, we have the following theorem:

Theorem 1.3. *The following property holds almost surely. Take one sample of lamination $L_\infty(\omega)$, the associated random process $\mathscr{M}_\infty(\omega)$ codes the lamination in the following way: $L_\infty(\omega)$ is the union of the chord $[xy]$ for all $x, y \in \mathbb{S}_1$ such that*

$$\mathscr{M}_\infty(\omega)(x) = \mathscr{M}_\infty(\omega)(y) = \min_{z \in Arc(x,y)} \mathscr{M}_\infty(\omega)(z)$$

Moreover, L_∞ is almost surely maximal in the sense of inclusion, that is to say we cannot add any more a new chord to it without intersecting any existed chord.

The report is organized as following. 2 gives more definitions about the lamination, containing different notations like its height function, the set of mass and a dynamic of a recursive lamination to describe its configuration. 3 will introduce the fragmentation theory and this is the main tool used to treat the study of recursive lamination. In this section, we will see also the proof of 1.1. In 4, we study how to code L_∞ by $\mathscr{M}_\infty$ by combing the result of fragmentation theory and studying an interesting branching process. Some further proprieties about this branching process will be stated in appendix.

Some other results are stated in **??**. Some various versions of lamination theory are included in appendix.

This report is mainly based on the work [1] and [3].

2 Random geodesic lamination

This section will talk about the definition of lamination, its maximality, a simple example *figela* and how to construct a lamination by a continuous function. In the end, we will give lamination a dynamic to add the chord in continuous time as a Markov process, to which we can embed the discrete recursive lamination defined at the first of 1. The importance of its dynamic is that it relates the fragmentation process stated later in 3.

2.1 Model

We first give the definition of lamination and some other standard notations.

Definition 2.1 (Lamination)**.** We denote $\mathbb{D} = \{z \in \mathbb{C} || z| < 1\}$ and its closure $\overline{\mathbb{D}}$. For two distinct points $x, y \in \mathbb{S}_1$, we call **chord** of **feet** x, y the segment of line in $\overline{\mathbb{D}}$ and denote it by $[xy]$, while $]xy[$ means $[xy] \cap \mathbb{D}$. The degenerated situation $[xx]$ is written as $\{x\}$. We say that $[xy]$ and $[x'y']$ don't cross(intersect) when $]xy[\cap]x'y'[= \emptyset$.

A **lamination** L of $\overline{\mathbb{D}}$ is a closed subset L of $\overline{\mathbb{D}}$ which can be written of the union of a collection of noncrossing chords.

The most simple example of lamination is *figela*, short for **finite geodesic lamination**.

Definition 2.2 (figela)**.** We call a lamination L **figela** when it is a union of finite noncrossing chord. Let S be the pairs of feet(unordered)

$$S = \{\{x_1, y_1\}, \{x_2, y_2\}\}$$

Then the lamination is just the union of the chords formed by the pairs in S.

If $\sharp S = n$, then $\overline{\mathbb{D}} \setminus \cap_{i=1}^{n} [x_i y_i]$ has $n+1$ connected components and we call them **fragments of the *figela***. Each fragment R has its **mass**

$$m(R) = \lambda(R \cap \mathbb{S}_1)$$

where λ represents the uniform probability on $\mathbb{S}_1$.

Remark. We use S to denote the pairs of feet and L_S to denote the lamination. Since there is a natural bijection between the two, sometimes we abuse the use of the two and when we talk about the distribution of L_S, it always refers the distribution of S, which but also characterises the law of L_S.

The height function is a good method to describe the geometry of *figela* and it has triangle inequality so in fact after equivalence relationship, it defines a metric on $\mathbb{S}_1$ and generate a tree like structure on the disk.

Definition 2.3 (Height function)**.** Let $u, v \in \mathbb{S}_1 \setminus Feet(S)$. The height between u and v in S is the number of chords of S crossed by the chord $[uv]$

$$H_S(u, v) = \sharp\{x, y \in S | [xy] \cap [uv] \neq \emptyset\}$$

Proposition 2.1 (Inequality of triangle of heigh function)**.** *Let S be a figela. For every $u, v, w \in S \setminus Feet(S)$, we have*

$$H_S(u, v) \leq H_S(u, w) + \ H_S(v, w)$$

Proof. We will give a closed form of height function and use it to prove the inequality. For and pair $x, y, u, v \in \mathbb{S}_1$, $]xy[\cap]uv[\neq \emptyset$ if and only if

$$\begin{aligned} f_{xy}(u) &= \langle u - x, (y-x)^{\perp} \rangle \\ f_{xy}(u) f_{xy}(v) &< 0 \end{aligned}$$

where $(y-x)^{\perp}$ is the a unit normal vector of $y - x$ and $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^2$. The choice of direction of the normal vector is not important, since the second formula just means that $[xy]$ separates two points and will not change sign by reversing the direction of the normal vector. Then

$$H_S(u, v) = \sum_{\{x,y\} \in S} \mathbf{1}_{f_{xy}(u) f_{xy}(v) < 0}$$

By considering the sign of function we obtain

$$\begin{aligned} H_S(u,v) &= \sum_{\{x,y\} \in S} \mathbf{1}_{f_{xy}(u) f_{xy}(v) < 0} \\ &= \sum_{\{x,y\} \in S} \mathbf{1}_{f_{xy}(u) f_{xy}(v) < 0} (\mathbf{1}_{f_{xy}(u) f_{xy}(w) < 0} + \mathbf{1}_{f_{xy}(w) f_{xy}(v) < 0}) \\ &= \sum_{\{x,y\} \in S} \mathbf{1}_{f_{xy}(u) f_{xy}(v) < 0} \mathbf{1}_{f_{xy}(u) f_{xy}(w) < 0} + \sum_{\{x,y\} \in S} \mathbf{1}_{f_{xy}(u) f_{xy}(v) < 0} \mathbf{1}_{f_{xy}(w) f_{xy}(v) < 0} \\ &\leq H_S(u,w) + H_S(w,v) \end{aligned}$$

□

Thanks to this triangle inequality, we can define a tree associate to *figela*.

Proposition 2.2 (Tree associated to *figela*)**.** *We define an equivalence relation for every $u, v \in \mathbb{S}_1 \setminus Feet(S)$*

$$u \simeq v \text{ if and only if } H_S(u, v) = 0$$

Then H_S define a metric on

$$\mathcal{T}_S = (\mathbb{S}_1 \setminus Feet(S)) / \simeq$$

In fact this coincides a metric on a tree.

Proof. By the argument of recurrence, we claim that in the case $\sharp S = n$, the number of fragments is $n + 1$. We construct the tree explicitly. We construct a non-oriented graph $G = (V, E)$ where

$$\begin{aligned} V &= \{R_i, |\text{fragments of S}\} \\ E &= \{\{R_i, R_j\} | R_i, R_j \text{are adjacent}\} \end{aligned}$$

Then an observation is that

$$\sharp E = \text{ number of chords } = n$$

since the adjacency of fragments means a chord between and a chord can be only the frontier of one couple. Then the the graph G is a tree and we denote it as $\mathcal{T}_S$.

Afterwords, we have to show that two definitions construct the same object. It is clear that the equivalence relation coincides the definition of fragment, so there is a bijection between the vertex of tree and the element in equivalence relation. Moreover, the definition of H_S coincides also with the distance between two fragments, that is the times to cross the chords, so the tree and $(\mathbb{S}_1 \setminus Feet(S))/\simeq$ have a bijection and the definition above gives a structure of tree. □

Remark. We see clearly that in the tree like structure $\mathcal{T}_S$, each vertex is the fragment of *figela* and the distance is the one between different fragments. This is a interesting structure and the reader may wonder if it will describe the structure when n goes to infinite? Then answer is partially correct. The height function is the key but it will be replaced by other function considered as the height function after some type of normalisation or scaling. The tree structure, although interesting and codes other models like **stable lamination** and **Brownian triangulation**, is not the center of recursive lamination and how to use it to describe the recursive lamination remain to study. In order to avoid confusing some similar notations, we keep the story of Brownian triangulation and stable lamination in appendix.

2.2 Maximal lamination

Definition 2.4 (Maximal lamination)**.** A lamination L is **maximal** if it is maximal for the inclusion relation among lamination of $\overline{\mathbb{D}}$.

Lemma 2.1 (Maximal lamination equals triangulation of $\overline{\mathbb{D}}$)**.** *A lamination L is maximal if and only if every connected component of $\overline{\mathbb{D}} \setminus L$ is an open triangle whose vertices belong to $\mathbb{S}_1$.*

Proof. Let L be a maximal lamination and we prove that all its fragments are open triangles. We suppose that it isn't the case, then it has at least one fragment who contains a qua-dragon and we can add one chord to build a bigger lamination and this contradicts the maximality.

For a lamination whose every fragment is an open triangle, when we choose two points on $\mathbb{S}_1$, in any case, the chord generated by them will cross or overlap another chord, so it is maximal. □

2.3 Coding a lamination by a continuous function

In this subsection, we introduce another method to generate a lamination by a continuous function. Different from the *figela*, the lamination generated by this method always have infinite chords. We can consider it as an alternative way to build height function after normalisation when n goes to infinite, because when n goes to infinite, almost surely the value of height function goes to infinite and loss its meaning.

Obviously, the function associated to the lamination should follow some properties, so we define at first this continuous function. We call it **contour function** since it has closed connection between $\mathbb{R}-$tree in appendix.

Definition 2.5 (Contour function)**.** A function g is called **contour function** if $g : [0,1] \to \mathbb{R}_+$ is continuous and $g(0) = g(1) = 0, g(x) > 0$ for $\forall x \in]0,1[$.

We recall the definition of pseudo-distance, who satisfies all the definition of distance except for two distinct point, the distance between them can be 0. A contour function defines a pseudo-distance on $[0,1]$ and this property is the base of the lamination coded by contour function.

Proposition 2.3 (Pseudo-distance defined by contour function)**.** *A contour function defines a pseudo-distance on* $[0,1]$ *by*

$$d_g(s,t) = g(s) + g(t) - 2 \min_{r\in[s\wedge t, s\vee t]} g(r)$$

An equivalence relation is also defined as $\overset{g}{\sim}$ *if and only if* $d_g(s,t) = 0$*, or* $g(s) = g(t) = \min_{r\in[s\wedge t, s\vee t]} g(r)$*.*

Proof. The symmetry and positivity are direct from the formula. We check the triangle inequality. We write

$$\begin{aligned} d_g(s,t) &= g(s) + g(t) - 2 \min_{r_1\in[s\wedge t, s\vee t]} g(r_1) \\ d_g(s,p) &= g(s) + g(p) - 2 \min_{r_2\in[s\wedge p, s\vee p]} g(r_2) \\ d_g(p,t) &= g(p) + g(t) - 2 \min_{r_3\in[p\wedge t, p\vee t]} g(r_3) \end{aligned}$$

We would like to prove $d_g(s,t) \le d_g(s,p) + d_g(p,t)$ and treat two cases $(1) s \le p \le t$ and $(2) s \le t \le p$ separately and the other situations are similar.

We recall a useful identity

$$a \wedge b + a \vee b = a + b$$

In the first case $s \le p \le t$,

$$\begin{aligned} d_g(s,t) &= g(s) + g(t) - 2 \min_{r_1\in[s\wedge t, s\vee t]} g(r_1) \\ &= g(s) + g(t) - 2 \min_{r_2\in[s\wedge p, s\vee p]} g(r_2) \wedge \min_{r_3\in[p\wedge t, p\vee t]} g(r_3) \\ &\le g(s) + g(t) - 2 \min_{r_2\in[s\wedge p, s\vee p]} g(r_2) \wedge \min_{r_3\in[p\wedge t, p\vee t]} g(r_3) \\ &\quad + 2d_g(p) - 2 \min_{r_2\in[s\wedge p, s\vee p]} g(r_2) \vee \min_{r_3\in[p\wedge t, p\vee t]} g(r_3) \\ &= d_g(s,p) + d_g(p,t) \end{aligned}$$

In the second case $s \le t \le p$,

$$\begin{aligned} d_g(s,t) &= g(s) + g(t) - 2 \min_{r_1\in[s\wedge t, s\vee t]} g(r_1) \\ &\le g(s) + g(t) - 2 \min_{r_1\in[s\wedge t, s\vee t]} g(r_1) \wedge \min_{r_3\in[p\wedge t, p\vee t]} g(r_3) \\ &\quad + 2d_g(p) - 2 \min_{r_3\in[p\wedge t, p\vee t]} g(r_3) \\ &= d_g(s,p) + d_g(p,t) \end{aligned}$$

Thus we finish the proof.

The triangle inequality implies directly the equivalence: suppose $r \overset{g}{\sim} s, s \overset{g}{\sim} t$, thus

$$0 \le d_g(r,t) \le d_g(r,s) + d_g(s,t) \le 0 \Longrightarrow d_g(r,t) = 0$$

□

Then, we give the construction of a lamination coded by a contour function.

Proposition 2.4 (Lamination coded by contour function)**.** *We denote $cl_g(s)$ the equivalence class of s with respect of to the equivalence relation $\overset{g}{\sim}$. We set $s \overset{g}{\approx} t$ when at least one of the two following condition is satisfied:*

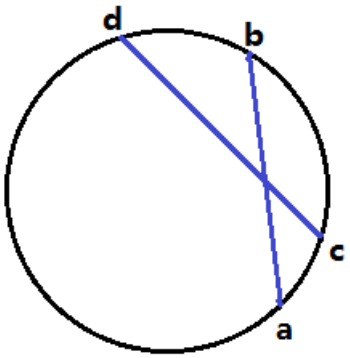
- $s \overset{g}{\sim} t, g(r) > g(s), \forall r \in [s \wedge t, s \vee t]$
- $s \overset{g}{\sim} t, s \wedge t = \min cl_g(s), s \vee t = \max cl_g(s)$

Then we set

$$L_g = \bigcup_{s \overset{g}{\approx} t} [e^{2\pi s} e^{2\pi t}]$$

is a lamination called ***the lamination coded by the function g****. It has the following two properties:*
(1)L_g is a closed.
(2)L_g is maximal if and only the local minima are distinct.

Remark. Generally, $\overset{g}{\approx}$ isn't a equivalence relation except in the maximal case.



Proof. We prove by contradiction that the construction is a lamination. We suppose that there exists two chords $]ab[\cap]cd[\neq \emptyset$. Then if they all meet the first condition of $\overset{g}{\approx}$, then it implies that all the values of g on the arc between the chord is strictly bigger than the value of the endpoint. That is $g(a) = g(b) < g(c)$ and $g(c) = g(d) < g(b)$ which is a contradiction.

In the second case, one of the pair, for example (a, b) satisfies the second condition of $\overset{g}{\approx}$. However, the crossing reveals another equivalence class outside the pair (a, b). Therefore, it is also a contradiction.

(1)We prove that the graph is closed. That is, if $s_n \overset{g}{\approx} t_n, s = \lim_{n\to\infty} s_n, t = \lim_{n\to\infty} t_n$, then we would like to prove $s \overset{g}{\approx} t$. Since pseudo-distance function d_g is continuous,

$$d_g(s_n, t_n) = 0 \Longrightarrow d_g(s, t) = 0$$

which implies that $s \overset{g}{\approx} t$. Without loss of generality, we suppose that $s < t$. Therefore, $\forall x \in]s, t[, g(x) \geq g(s) = g(t)$.

We still prove by contradiction. We suppose that neither of the condition of $\overset{g}{\approx}$ above are valid, then there exist r such that

$$s < r < t, g(r) = g(s) = g(t) = \min_{x \in [s,t]} g(x)$$

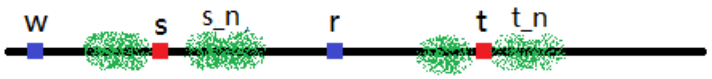
and there exists, another equivalence class outside $]s, t[$, for example

$$w < s < t, d_g(w, s) = 0$$

Then, we analyse s_n, t_n. Since s, t are respective two limit of the series, $\exists s_n, t_n$ such that

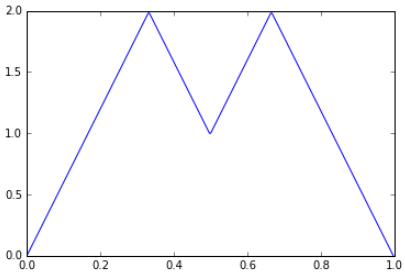
$$w < s_n < r < t_n, s_n \overset{g}{\approx} t_n$$

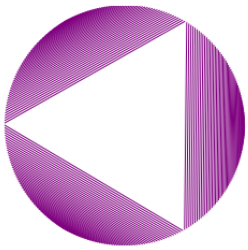
Then



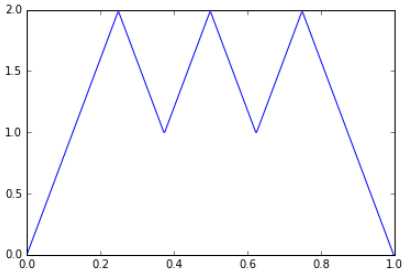
$$\begin{aligned} d_g(s_n, t_n) = 0, r \in]s_n, t_n[&\implies g(s_n) \leq g(r) \\ d_g(w, r) = 0, s \in]w, r[&\implies g(s_n) \geq g(r) \end{aligned}$$

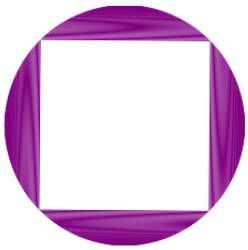
Therefore, $s_n, t_n \in cl_g(s)$, but in $]s_n, t_n[$ there is r which achieve the minimum and outside them there is w which is also the equivalence. That is a contradiction. So our supposition is not correct and we have $s \overset{g}{\approx} t$.





(2)Then we prove the equivalent condition of maximality. The full proof can be found in [4] and here we just give a quick review. First, let g be a contour function having distinct local minima, then for every value, it has at most three points which attain it so they form a triangle following the construction of lamination. Thus, each fragment has at most 3 angles and the lamination is maximal by the 2.1.





On the other hand, if there is a local minima attained at least twice i.e $\exists 0 < P_1 < P_2 \cdots < P_m < 1$ and $g(P_1) = g(P_2) \cdots = g(P_m)$ are all local minima. Then, considering that g are strictly positive except $0, 1$, there exist two other points

$$P_0 = \sup_{t \in (0, P_1)} \{t | g(t) = g(P_1)\} \tag{1}$$

$$P_{m+1} = \inf_{t \in (P_m, 1)} \{t | g(t) = g(P_m)\} \tag{2}$$

The condition of local minima implies that there are chords $[P_0P_1], [P_1P_2] \dots [P_mP_{m+1}]$ but no chord $[P_0P_2]$ nor $[P_1P_{m+1}]$ which contradicts the maximality. □

2.4 Recursive lamination: discrete time and continuous time

In this subsection, we introduce how to give the lamination a dynamic in continuous time and draw connection between this model with the original one. This new model in continuous time can be considered as an example of fragmentation process, which we will talk about in the next section.

Before stating the construction of our new model, we do some calculus about the model L_n.

Proposition 2.5. *Let L_n be a lamination of n chords and $n+1$ fragments $R_1^n, R_2^n \dots R_{n+1}^n$, then the waiting time to add one new chord follows the geometric law of parameter $\sum_{i=1}^{n+1} m(R_i^n)^2$. Given a new chord is added, the probability that it is added on R_k^n has a probability $\frac{m(R_k^n)^2}{\sum_{i=1}^{n+1} m(R_i^n)^2}$.*

Proof. The proof is simple. We just do calculus of probability:

$$\mathbb{P}(]UV[\cap L_n = \emptyset) = \mathbb{P}(\exists i, \text{ s.t } U, V \in R_i^n) = \sum_{i=1}^{n+1} m(R_i^n)^2$$

$$\mathbb{P}(U, V \in R_k^n |]UV[\cap L_n = \emptyset) = \frac{\mathbb{P}(U, V \in R_k^n)}{\mathbb{P}(\exists k, \text{ s.t } U, V \in R_k^n)} = \frac{m(R_k^n)^2}{\sum_{i=1}^{n+1} m(R_i^n)^2}$$

□

This proposition inspires us to design a continuous model: since a continuous process with the property of loss of memory like geometric law is a Poisson process and if it also has information of configuration, we should make it a Markov jump process, we propose the definition as following.

Definition 2.6 (α self-similar *figela* process)**.** We define random element value in *figela* space

$$S_\alpha(t) : \Omega \to \textit{figela}$$

and $S_\alpha(t)$ a Markov jump process, which takes jump at the point $0 = \tau_0 < \tau_1 < \tau_2 \dots$ and takes constant values in interval $[\tau_n, \tau_n + 1[$.

Filtration:We define the filtration as

$$\mathcal{F}_n = \sigma(\tau_0, \tau_1 \dots, \tau_n, S_\alpha(\tau_0), S_\alpha(\tau_1) \dots S_\alpha(\tau_n))$$

Law of jump time:We denote $R_j^n, 1 \le j \le n+1$ fragments of $S_\alpha(\tau_n)$ and $(e_j)_{1 \le j \le n+1} n+1$ independent random variables of exponential law of parameter 1 and

$$\begin{aligned} \mathcal{E}_j &= m(R_j^n)^{-\alpha} e_j \\ \mathcal{E}_{j_0} &= \min\{\mathcal{E}_j, 1 \le j \le n+1\} \\ \tau_{n+1} &= \tau_n + \mathcal{E}_{j_0} \end{aligned}$$

then $\mathcal{E}_{j_0}$ follows the exponential law of parameter $\sum_{j=1}^{n+1} m(R_j^n)^\alpha$.

Law of transition:Let j_0 be the index which $\mathcal{E}_j$ obtains the minimum, then Let X_{n+1}, Y_{n+1} be uniformly distributed on $R_{j_0}^n \cap \mathbb{S}_1$ and

$$S_\alpha(\tau_0) = \emptyset, S_\alpha(\tau_{n+1}) = S_\alpha(\tau_n) \cup \{X_{n+1}, Y_{n+1}\}$$

then X_{n+1}, Y_{n+1} are all on R_k^n with probability $\frac{m(R_k^n)^\alpha}{\sum_{j=1}^{n+1} m(R_j^n)^\alpha}$.

Remark. In fact, we have to check some proprieties in the definition. The **Markov property** follows that the law of jump time and the law of transition only depend on the state $(S_\alpha(\tau_n), \tau_n)$. Then, we check the two claims about the law of $\mathcal{E}_{j_0}$ and X_{n+1}, Y_{n+1}:

$$\begin{aligned} \mathbb{P}(\mathcal{E}_{j_0} > t) &= \mathbb{P}(\forall j, \mathcal{E}_j > t) = \Pi_{j=1}^{n+1} \mathbb{P}(\mathcal{E}_j > t) = \exp(\sum_{j=1}^{n+1} m(R_j^n)^\alpha) \\ \mathbb{P}(X_{n+1}, Y_{n+1} \in R_k^n) &= \int_0^\infty [\Pi_{j \ne k} \mathbb{P}(\mathcal{E}_j > t)] m(R_k^n)^\alpha e^{-m(R_k^n)^\alpha t} dt \\ &= \int_0^\infty m(R_k^n)^\alpha e^{-\sum_{j=1}^{n+1} m(R_j^n)^\alpha t} dt \\ &= \frac{m(R_k^n)^\alpha}{\sum_{j=1}^{n+1} m(R_j^n)^\alpha} \end{aligned}$$

Remark (Another definition of α self-similar *figela* process). Another interpretation about this process can be stated as following: We start from a $S_\alpha(0) = \emptyset$. After each time of adding a new chord, every fragment R evolves independently and has lifetime as a random variable which follows the exponential law of parameter $m(R)^\alpha$. When the it "dies", if will be divided into two fragments by

adding a new chord in it whose two endpoints are chosen independently and uniformly on R. This description is just the definition except for the way to define τ_n. However, the property of **loss of memory** implies that

$$\mathbb{P}(\mathcal{E}_j > t+s | \mathcal{E}_j > s) = \mathbb{P}(\mathcal{E}_j > t)$$

so in fact the waiting time is also same.

Then we state the connection between the α self-similar *figela* process and the process L_n and especially in the case $L_{S_2(t)}$. Here we use $L_{S_2(t)}$ to denote the lamination coded by the pair of feet set $S_2(t)$. By the calculus above, we observe that the waiting time for L_n and $L_{S_2}(t)$ are all $\frac{1}{\sum_{j=1}^{n+1} m(R_j^n)^2}$ and the probability to add to each fragment is also same, given that they have the same law of fragment. However, here for L_n, the number of chords may be less than $n+1$ but for $L_{S_{\tau_n}}$ it is necessarily equal. Therefore, the key idea is an embedding from L_n to $L_{S_2(n)}$ thanks to another *figela* process we will define as following.

Definition 2.7 (Poisson *figela* process)**.** We define a Poisson point measure on space $R_+ \times \mathbb{S}_1 \times \mathbb{S}_1$ equipped with a measure $dt \otimes \lambda \otimes \lambda$ where dt represents Lebesgue measure and λ just the uniform probability on circle.

Then we note $\{(t_m, x_m, y_m)\}_{m \geq 1}$ the value and generate a *figela* from it. We let $\mathscr{S}$ be the set of pairs of feet:

$$\mathscr{S}(t) : \Omega \to \textit{figela}, \mathscr{S}(0) = \emptyset, \text{constant on} [t_m, t_{m+1}[$$

Moreover, we follow the recurrence: if $L_{\mathscr{S}(t_m)} \cap]x_{m+1} y_{m+1}[= \emptyset$, then $\mathscr{S}(t_{m+1}) = \mathscr{S}(t_m) \cup \{x_{m+1}, y_{m+1}\}$, otherwise $\mathscr{S}(t_{m+1}) = \mathscr{S}(t_m)$.

Proposition 2.6 (Embedding from L_m to $L_{S_2(t)}$)**.** *(1)We adapt the notation of Poisson figela and α self-similar figela process and L_n, then*

$$L_m \overset{(d)}{=} L_{\mathscr{S}(t_m)}, \lim_{n \to \infty} \frac{t_m}{m} \overset{a.s}{=} 1$$

(2)Moreover, if we define that

$$\tilde{t}_0 = 0, \tilde{t}_n = \inf\{t_m | \mathscr{S}(t_m) \neq \mathscr{S}(\tilde{t}_{n-1})\}$$

then we have

$$\tilde{t}_n \overset{(d)}{=} \tau_n, L_{S_2(\tau_n)} \overset{(d)}{=} L_{\mathscr{S}(\tilde{t}_n)}$$

(3)If we suppose that $\frac{N(L_{S_2(\tau_n)})}{\sqrt{\tau_n}}$ converge in a limit, we have

$$\lim_{m \to \infty} \frac{N(L_m)}{\sqrt{m}} \overset{(d)}{=} \lim_{n \to \infty} \frac{N(L_{S_2(\tau_n)})}{\sqrt{\tau_n}}$$

Proof. (1)We let R be a fragment on $\mathbb{S}_1$, then by using the definition of Poisson point measure

$$\sharp\{(t_m, x_m, y_m) \in [0, t[\times R \times R\} \overset{(d)}{\sim} \text{Poisson}(m(R)^2 t)$$

and the same argument gives the law that

$$\sharp\{(t_m, x_m, y_m) \in [0, t[\times \mathbb{S}_1 \times \mathbb{S}_1\} \overset{(d)}{\sim} \text{Poisson}(t)$$

These properties gives an equivalent description given by Poisson process: We take the jump point $t_1, t_2 \dots$ by Poisson process with density 1 and each time choose a couple of points $(x_1, y_1), (x_2, y_2) \dots$ independently and uniformly on the circle.

By this construction, we get immediately that the conditional law of L_m and $L_{\mathscr{S}(t_m)}$ are same since they take the same random dynamic. Given they have the same initial law, we get $L_m \overset{(d)}{=} L_{\mathscr{S}(t_m)}$. The convergence of jump time follows the law of large number. In fact, the description of Poisson process shows that $t_m = \sum_{i=1}^{m} \xi_i, \xi_i \overset{(d)}{\sim} \exp(1)$i.i.d, then

$$\lim_{m\to\infty} \frac{t_m}{m} \overset{(a.s)}{=} \mathbb{E}(\xi_1) = 1$$

(2) The second part takes the same argument. This time $\tilde{t}_n$ means exactly the first time that two points are chosen in the same fragment and follows the same description of 2 self-similar *figela* process. An intuitive interpretation is that, the 2 self-similar *figela* is a process of Poisson *figela* after erasing all the attempt to add chord but fail. Therefore, we establish the identity of law.

(3) Finally we prove the convergence under hypothesis:

$$\forall t_m \in [\tilde{t}_{n-1}, \tilde{t}_n[, \frac{N(L_{S_2(\tau_n)}) - 1}{\sqrt{\tau_n}} \leq \frac{N(L_{\mathscr{S}(t_m)})}{\sqrt{t_m}} \overset{(d)}{=} \frac{N(L_{S_2(\tau_{n-1})})}{\sqrt{t_m}} \leq \frac{N(L_{S_2(\tau_{n-1})})}{\sqrt{\tau_{n-1}}}$$
$$\Longrightarrow \lim_{m\to\infty} \frac{N(L_{\mathscr{S}(t_m)})}{\sqrt{t_m}} \overset{(d)}{=} \lim_{n\to\infty} \frac{N(L_{S_2(\tau_n)})}{\sqrt{\tau_n}}$$

Moreover, we have

$$\lim_{m\to\infty} \frac{N(L_{\mathscr{S}(t_m)})}{\sqrt{t_m}} \overset{(d)}{=} \lim_{m\to\infty} \frac{N(L_m)}{\sqrt{m}} \frac{\sqrt{m}}{\sqrt{t_m}} \overset{(d)}{=} \lim_{m\to\infty} \frac{N(L_m)}{\sqrt{m}}$$

We conclude that

$$\lim_{m\to\infty} \frac{N(L_m)}{\sqrt{m}} \overset{(d)}{=} \lim_{n\to\infty} \frac{N(L_{S_2(\tau_n)})}{\sqrt{\tau_n}}$$

□

This embedding proposition helps us carry the study of discrete recursive lamination to 2 self-similar *figela* process, which makes it possible to use the theory of fragmentation theory.

3 Random Fragmentation

In this section, we will talk about the random fragmentation theory. An intuitive introduction of the fragmentation theory can be put in the situation of the division of the particles. Suppose that there is a type of particle, its lifetime follows the exponential law in parameter of the power α of its mass and when it dies, it will divide into two halves and the proportion of each offspring follows a law ν. All the particles evolve identically and independently.

The reader may find that the notation of fragmentation theory is so closed to that of random recursive lamination, that explains why we draw the connection between the two. However, the difficulty is sometimes to find a good law ν and power α so that it describes the model we need. Then, the theoretic result like convergence of martingale can be applied to the random recursive lamination model.

3.1 Model

In this subsection, we give the definition of the fragmentation process and a tree-like structure to describe this process.

Definition 3.1 (Binary fragmentation process)**.** We continue the description of the evolution of a particle system in a special situation: each particle has its **mass** and the state of system can be described by a decreasing real-valued sequence

$$\mathcal{S}^{\downarrow} = (s_1, s_2, \dots), 1 \geq s_1 \geq s_2 \geq \dots$$

where s_i represents the mass of particles. Therefore a binary fragmentation process is a Markov random process $X^{(\alpha)}(t)$ who takes value on $\mathcal{S}^{\downarrow}$:

$$X^{(\alpha)}(t) : \Omega \longrightarrow \mathcal{S}^{\downarrow}, X^{(\alpha)}(t) = (s_1(t), s_2(t), \dots)$$

The particle of mass s_i has its **lifetime**, a random variable of exponential law of parameter of $(s_i(t))^{\alpha}$. When it dies, it divide into two particles and we call their distribution of mass **disloaction measure**:

$$\begin{array}{lll} \nu & \text{value on set} & \{(\xi_0, \xi_1) | 1 > \xi_0 \geq \xi_1 \geq 0\} \\ & & \nu(\xi_1 > 0) > 0 \\ & & \nu(\xi_0 = 0) = 0 \end{array}$$

That is, the two offspring of $s_i(t)$ have respectively mass of $\xi_0 s_i(t), \xi_1 s_i(t)$ and the distribution of ξ_0, ξ_1 is sampled independently by ν of the past. The two additional properties make this process non-trivial. When $\xi_0 + \xi_1 = 1$, we call the process **conservative**, otherwise **dissipative**.

Finally, we recall that all the particle evolve independently. We can also define the term **total mass of power** p by $\sum_{i=1}^{\infty}(s_i(t))^p$.

Given a fragmentation chain, it's nature to equip it with a genealogical structure. This notation describes well the evolution of system by the relationship of ancestor and offspring.

Definition 3.2 (Generation tree)**.** We define

$$\mathbb{T} = \bigcup_{k \geq 0} \{0, 1\}^k$$

$\{0,1\}^0 = \emptyset$. Then, if the $X^{(\alpha)}(0)$ contains only one particle, each particle can be marked by an elements in $\mathbb{T}$.(If it starts with several particles, it should be a forest or several copies of $\mathbb{T}$). The first particle is marked by $\emptyset$. By recurrence, the two offspring of particle $u \in \mathbb{T}$ is $u0, u1$. Every element should have its ancestor, so at instant t if $u_i, i \in \{0,1\}$ exists, u should also be present. We also use

$$\mathbb{T}_n = \bigcup_{0 \le k \le n} \{0,1\}^k$$

to represent the evolution until the generation n.

We can mark the particle by triple (s_u, a_u, ζ_u), which represents its mass, time of birth, and lifetime. Obviously, the first particle is born at time 0 with a given mass. Then, we note $\{e_u\}_{u \in \mathbb{T}}$ a family of $i.i.d$ random variable of exponential law of parameter 1 and (ξ_{u0}, ξ_{u1}) defined by the dislocation measure ν. The definition of binary fragmentation process is:

$$\forall i \in \{0,1\}, u \in \mathbb{T}, s_{ui} = s_u \xi_{ui}, a_{ui} = a_u + \zeta_u, \zeta_u = (s_u)^{-\alpha} e_u$$

At the end of this subsection, we state the property of scaling of this process.

Proposition 3.1 (Scaling property)**.** *Let $X^{(\alpha)}(t)$ be the standard binary fragmentation process of dislocation measure ν starting with a particle of unit mass and $X_M^{(\alpha)}(t)$ be the one of the same dislocation measure starting with mass M, then*

$$X_M^{(\alpha)}(t) \overset{(d)}{=} M X^{(\alpha)}(M^\alpha t)$$

The proof is simple, we just check that they have the same initial condition, law of lifetime and dislocation law. This law is very important since it has some further generalized versions.

3.2 Malthusian exponent and key martingales

In this subsection, we will introduce Malthusian exponent and several important martingales of fragmentation process. These martingales provide the powerful tools to treat the evolution of the process.

We review the definition of dislocation measure. In the dissipative condition, $\xi_0 + \xi_1 < 1$ so the usual total mass(total mass of power 1) is drained. We would like to find a power to modify it conservative, that inspire the definition of Malthusian exponent.

Definition 3.3 (Malthusian exponent)**.** For every $p > 0$ we define

$$\kappa_\nu(p) = \int_{[0,1]^2} (1 - y_0^p - y_1^p) \nu(dy_0, dy_1)$$

Since $\kappa_\nu(0) = -1, \kappa_\nu(+\infty) = 1$, it exists unique exponent p^* such that

$$\kappa_\nu(p^*) = 0$$

We call it **the Malthusian exponent of** ν.

Theorem 3.1 (Martingale)**.** *Given a binary fragmentation process of index α $X^{(\alpha)} = (s_1^{(\alpha)}, s_2^{(\alpha)} \dots)$, for every $t \geq 0$ and $\alpha > 0$ then*

(1)We define a process

$$\mathscr{M}^{(\alpha)}(t) = \sum_{i=1}^{\infty} (s_i^{(\alpha)}(t))^{p*}, t \geq 0$$

is a uniformly integrable martingale and converges almost surely to a limiting random variable $\mathscr{M}_\infty$ called ***intrinsic martingale****, which does not depend on α. For all $q \geq 1, \mathscr{M}_\infty^{\shortparallel} < \infty$. It also has an identity : for (Σ_1, Σ_2) following the law given by dislocation measure ν, we have*

$$\mathscr{M}_\infty \overset{(d)}{=} \Sigma_1^{p*} \mathscr{M}'_\infty + \Sigma_2^{p*} \mathscr{M}''_\infty$$

where $\mathscr{M}'_\infty, \mathscr{M}''_\infty$ are two identical independent copies of $\mathscr{M}_\infty$.

(2)For every real $p \geq 0$, the process

$$e^{t\kappa_\nu(p)} \sum_{i=1}^{\infty} (s_i^{(0)}(t))^p, t \geq 0$$

is a martingale and converges a.s to a positive limiting random variable.

(3)Let $\alpha > 0$. Assume that $\int s_2^{-\alpha} \nu(ds_1, ds_2)$ for some α. Then for every $p \geq 0$,

$$t^{(p-p*)/\alpha} \sum_{i=1}^{\infty} (s_i^{(\alpha)}(t))^p \xrightarrow{L^2} K_\nu(\alpha, p) \mathscr{M}_\infty$$

where $K_\nu(\alpha, p)$ is a constant only depending on α, p, ν and $\mathscr{M}_\infty$ is the limit of intrinsic martingale.

Proof. We give an intuitive justification for the theorem. For (1), since $\mathscr{M}^{(\alpha)}(t)$ only changes after one jump time, we have only to check for jump time $t_1, t_2, t_3 \dots$ that

$$\mathbb{E}[\mathscr{M}^{(\alpha)}(t_{n+1}) | \mathscr{M}^{(\alpha)}(t_n)] = \mathscr{M}^{(\alpha)}(t_n)$$

In fact, suppose that at time t_{n+1}, $s_i(\alpha)(t_{n+1})$ is chosen and divided into $s_{i0}^{(\alpha)}(t_{n+1}), s_{i1}^{(\alpha)}(t_{n+1})$, then

$$\begin{aligned}
\mathbb{E}[\mathscr{M}^{(\alpha)}(t_{n+1}) | \mathscr{M}^{(\alpha)}(t_n)] &= \mathbb{E}[\mathscr{M}^{(\alpha)}(t_n) + (s_{i0}^{(\alpha)}(t_{n+1}))^{p*} + (s_{i1}^{(\alpha)}(t_{n+1}))^{p*} - (s_i^{(\alpha)}(t_{n+1}))^{p*} | \mathscr{M}^{(\alpha)}(t_n)] \\
&= \mathscr{M}^{(\alpha)}(t_n) + (s_i^{(\alpha)}(t_{n+1}))^{p*} \int (y_0^{p*} + y_1^{p*} - 1) \nu(dy_0, dy_1) \\
&= \mathscr{M}^{(\alpha)}(t_n)
\end{aligned}$$

For(2), we use 3.1 and notice that the lifetime doesn't depend on the mass

$$X_M^{(0)}(t) \overset{(d)}{=} M X^{(0)}(t)$$

so after division, the particle follows the same law of evolution. If we define that

$$\mathbb{E}[\sum_{i=1}^{\infty} (s_i^{(0)}(t))^p] = \mathbb{E}[\sum_{i=1}^{\infty} (s_i^{(0)}(l))^p] A(l, t), \forall l \leq t$$

then the scaling property and Markov property imply that $A(s,t)$ is stationary so that

$$A(l,t) = A(0,t-l), A(0,l+t) = A(0,l)A(0,t)$$

Moreover, $A(0,t)$ can be calculated informally as

$$\frac{dA(0,t)}{dt} = \lim_{t\to 0} \frac{(1-e^t)\kappa_\nu(p)}{t} = -\kappa_\nu(p)$$

We get $A(0,t) = e^{-\kappa_\nu(p)t}$. Then

$$\begin{aligned}
\mathbb{E}[e^{t\kappa_\nu(p)} \sum_{i=1}^{\infty} (s_i^{(0)}(t))^p | \mathcal{F}_l] &= \mathbb{E}[e^{t\kappa_\nu(p)} A(l,t) \sum_{i=1}^{\infty} (s_i^{(0)}(l))^p | \mathcal{F}_l] \\
&= \sum_{i=1}^{\infty} (s_i^{(0)}(l))^p e^{-(t-l)\kappa_\nu(p)} e^{t\kappa_\nu(p)} \\
&= e^{\kappa_\nu(p)l} \sum_{i=1}^{\infty} (s_i^{(0)}(l))^p
\end{aligned}$$

this finishes the proof. □

3.3 Application 1: counting the number of chords

As we have stated in the last section, *figela* process can be considered as a type of fragmentation process. In fact, the description of the *figela* process is almost parallel with that of fragmentation process. So, in this subsection, we draw the connection between the two and answer the 1.1 by using 3.1.

Proposition 3.2 (*figela* process to fragmentation process)**.** *We recall the α self-similar figela process $S_\alpha(t)$ and note $R_1^\alpha(t), R_2^\alpha(t)\dots$ its fragment ordered by decreasing mass, then*

$$X_\alpha(t) = (m(R_1^\alpha(t)), m(R_2^\alpha(t))\dots)$$

is a fragmentation process with parameter (α, ν_C) where ν_C is defined as

$$\int_{[0,1]^2} F(y_0,y_1)\nu(dy_0,dy_1) = 2\int_{\frac{1}{2}}^1 F(u,1-u)du$$

Proof. It is easy to check this definition. The jump time of fragmentation process is exactly same as that one of *figela* process. The dislocation measure $\nu(y_0,y_1)$ describes the fact that each time of division, the fragment into two parts with mass uniformly distributed. □

Then, we apply the 3.1 and obtain directly

Proposition 3.3. *(1)If $\alpha = 0$,*

$$e^{-t}\sharp S_0(t) \xrightarrow[n\to\infty]{a.s} \mathcal{E}$$

where $\mathcal{E}$ is a random variable of exponential law of parameter 1.

(2)If $\alpha > 0$, we have

$$t^{-1/\alpha}\sharp S_\alpha(t) \xrightarrow[n\to\infty]{a.s} \frac{\Gamma(1/\alpha)}{\Gamma(2/\alpha)}$$

Proof. For (1), we use (2) of 3.1 in the case $p = 0$.

For (2), we use (3) of 3.1 in the case $p = 0$.

Law? Constant?

□

Recall the 2.6, we get the answer about the asymptotic number of chord in L_n.

Corollary 3.1 (Number of L_n)**.** *The number of chords in L_n has an asymptotic limit*

$$n^{-1/2}N(L_n) \xrightarrow[n\to\infty]{a.s} \sqrt{\pi}$$

Proof. Using 3.3 in the case $\alpha = 2$, we have

$$t^{-1/2}\sharp S_2(t) \xrightarrow[n\to\infty]{a.s} \frac{\Gamma(1/2)}{\Gamma(1)} = \sqrt{\pi}$$

Moreover, we recall 2.6 that the discrete model and the continuous one have the same asymptotic law:

$$\lim_{t\to\infty} \frac{N(L_n)}{\sqrt{n}} = \lim_{t\to\infty} \frac{\sharp S_2(t)}{\sqrt{t}} \overset{a.s}{=} \sqrt{\pi}$$

□

3.4 Application 2: studying the limit height function

In this subsection, we use fragmentation theorem to study the asymptotic behaviours of height function. This is less evident than the number of chords since it's hard observe the direct connection at first glance. However, inspired by the last application, we would like to look for a good measure to count only the mass of fragments intersected with the given chord, then we will follow the same strategy as counting the number of chord.

Firstly, we define a special type of fragment.

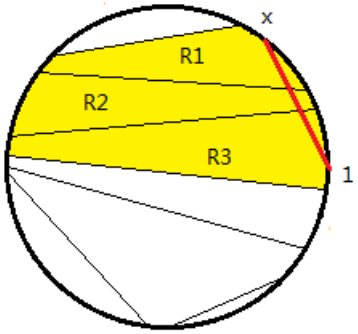
Definition 3.4 (Separating fragment)**.** Let S be the pair set of feet of *figela* and $x, y \in \mathbb{S}_1 \setminus Feet(S)$. We define the fragments of S which intersects the chord $[xy]$ separating fragment x from y. We range these fragments in decreasing mass so that they form an element in $S^{\downarrow}$. We denote them by

$$R^{x,y}(S), R^{x,y}(S), R^{x,y}(S) \dots$$

Notice that the height function is just the number of fragments minus 1

$$H_S(x, y) = \sharp\{ \text{ fragments intersect } [xy]\} - 1 = \sum_{i=1}^{\infty} m(R_i^{(x,y)}(S))^0 - 1$$

and it has a form similar in the 3.1. What we need to do is to construct a fragmentation process associated the separating fragment x from y. Here we give a construction and proof for a easy version, where $x = 1$ and $y = V$ a random variable uniformly distributed on $\mathbb{S}_1$. The proof for y a fixed point need more arguments and reader can check [1].



Proposition 3.4 (Separating fragmentation process). *Given a fragmentation process $(S_\alpha(t), t \geq 0, \alpha \geq 0)$ defined in 3.2 and V a random variable valued uniformly on $\mathbb{S}_1$ independent to $S_\alpha(t)$, then the sequence of fragments separating 1 from V in $S_\alpha(t)$ namely*

$$\mathcal{X}_\alpha(t) = (m(R_1^{1,V}(S_\alpha(t))), m(R_2^{1,V}(S_\alpha(t)))\dots)$$

is a fragmentation process with parameter (α, ν_D) where ν_D satisfies

$$\int_{[0,1]^2} F(y_1, y_0)\nu_D(dy_1, dy_0) = 2\int_0^1 u^2 F(u,0)du + 4\int_{\frac{1}{2}}^1 u(1-u)F(u,1-u)du$$

The proof of this proposition consists of two parts:
(1)The dislocation measure describes the law of counting only the fragments intersecting the chord $[1V]$.
(2)After each division, the each part evolves independently and also follows the way of counting the specified fragments, especially the lifetime, the dislocation measure.

Before proving the first part, we give an important bijection after the appearance of the first chord.

Proposition 3.5 (Bijection of two disks after the first chord). *Given a figela precess $S_\alpha(t)$ and τ the time of the appearance of the first chord, (a,b) the pair of feet so that $a = e^{2\pi i U_1}, b = e^{2\pi i U_2}$, where (U_1, U_2) has density $2 \cdot \mathbf{1}_{0<U_1\leq U_2<1}$ of Lebesgue measure. We denote also $M = 1-(U_1-U_2)$ the mass of the fragment containing 1. Then we have two mapping $\psi_{U_1,U_2} : [0,U_1]\cup[U_2,1] \to [0,1]$, $\phi_{U_1,U_2} : [U_1,U_2] \to [0,1]$*

$$\begin{aligned}
\psi_{U_1,U_2}(r) &= \begin{cases} \frac{r}{M} \text{ if } 0 \leq r \leq U_1 \\ \frac{r-(U_2-U_1)}{M} \text{ if } U_2 \leq r \leq 1 \end{cases} \\
\phi_{U_1,U_2}(r) &= \frac{r-U_1}{1-M} \text{ if } U_1 \leq r \leq U_2
\end{aligned}$$

These two mappings induce also the mapping from two fragments to two disks.

$$\begin{aligned}
\Psi_{a,b}(\exp(2\pi i r)) &= \exp(2\pi i \psi_{U_1,U_2}(r)) \\
\Phi_{a,b}(\exp(2\pi i r)) &= \exp(2\pi i \phi_{U_1,U_2}(r))
\end{aligned}$$

After the appearance of the first chord, we denote the two fragments R', R'' where R' contains 1 and the partition of $S_\alpha(t)$ on two halves $S_\alpha^{(R')}(t), S_\alpha^{(R'')}(t)$. Then conditionally on (τ, U_1, U_2) we have

$$((\Psi_{a,b}(S_\alpha^{(R')}(t+\tau)))_{t\geq 0}, (\Phi_{a,b}(S_\alpha^{(R')}(t+\tau)))_{t\geq 0}) \overset{(d)}{=} ((S'_\alpha(M^{\alpha t}))_{t\geq 0}, (S''_\alpha(1-M)^\alpha t)_{t\geq 0})$$

where the S'_α, S''_α are two independent copies of figela process.

Proof. After one division, the two fragments can be considered as two disks by checking the mapping well defined. They are independent by the definition of *figela* process. The scaling of time comes from the fact the lifetime depends on its mass. □

The first part of 3.4 can be proved directly by calculus.

Proof. **First part of 3.4** We first check the initial case ν_D describes the fact that we count only the fragments intersecting the chord $[1V]$. A sufficient and necessary condition that a fragment intersects $[1V]$ is that it contains at least one foot between 1 and V. Since R' contains necessarily the foot 1, it has two situations $V \in R'$ and $V \in R''$.

$$\begin{aligned}
&\int_{[0,1]^2} F(y_0, y_1)\nu_D(dy_0, dy_1) \\
= \;& 2\int_{[0,1]^3} \mathbf{1}_{V\in R''}\mathbf{1}_{U_2-U_1>1-(U_2-U_1)} F(U_2-U_1, 1-(U_2-U_1))dU_1dU_2dV \\
+ \;& 2\int_{[0,1]^3} \mathbf{1}_{V\in R''}\mathbf{1}_{U_2-U_1<1-(U_2-U_1)} F(1-(U_2-U_1), U_2-U_1)dU_1dU_2dV \\
+ \;& 2\int_{[0,1]^3} \mathbf{1}_{V\in R'} F(1-(U_2-U_1), 0)dU_1dU_2dV
\end{aligned}$$

By the change of variable $x = U_2 - U_1$, We simplify the equations, the first term is

$$\begin{aligned}
&2\int_{[0,1]^3} \mathbf{1}_{V\in R''}\mathbf{1}_{U_2-U_1>1-(U_2-U_1)} F(U_2-U_1, 1-(U_2-U_1))dU_1dU_2dV \\
= \;& 2\int_{[0,1]^2} \mathbf{1}_{x>\frac{1}{2}}\mathbf{1}_{0<U_1<1-x} xF(x, 1-x)dU_1dx \\
= \;& 2\int_{\frac{1}{2}}^1 x(1-x)F(x, 1-x)dx
\end{aligned}$$

Then the second term is

$$\begin{aligned}
&2\int_{[0,1]^3} \mathbf{1}_{V\in R''}\mathbf{1}_{U_2-U_1<1-(U_2-U_1)} F(1-(U_2-U_1), U_2-U_1)dU_1dU_2dV \\
= \;& 2\int_{[0,1]^2} \mathbf{1}_{x<\frac{1}{2}} xF(1-x, x)dU_1dx \\
= \;& 2\int_0^{\frac{1}{2}} x(1-x)F(1-x, x)dx \\
= \;& 2\int_{\frac{1}{2}}^1 x(1-x)F(x, 1-x)dx
\end{aligned}$$

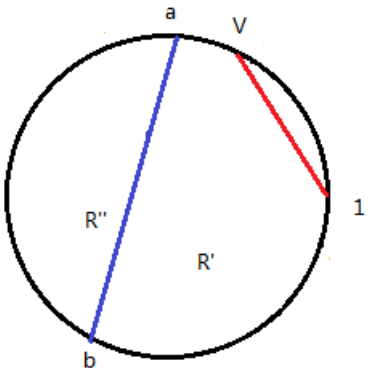
Finally, the third term is

$$\begin{aligned} & 2\int_{[0,1]^3} \mathbf{1}_{V\in R'} F(1-(U_2-U_1),0)dU_1dU_2dV \\ = \; & 2\int_{[0,1]^2} \mathbf{1}_{0<U_1<1-x}(1-x)F(1-x,0)dU_1dx \\ = \; & 2\int_0^1 (1-x)^2F(1-x,0)dx \\ = \; & 2\int_0^1 x^2F(x,0)dx \end{aligned}$$

However, this proves only that the dislocation measure describes the first division. We will prove that in the following process, each part also obeys this law of division. □

Then, we use the result from 3.5 to prove the second part of 3.4.

Proof. **Second part of 3.4** We continue to study the dislocation measure in the further process. It suffices to check that after the first chord, the division in two halves R', R'' follows the same law ant then we apply the recurrence to the further process. In the first case where V is on R'. Then, since we count



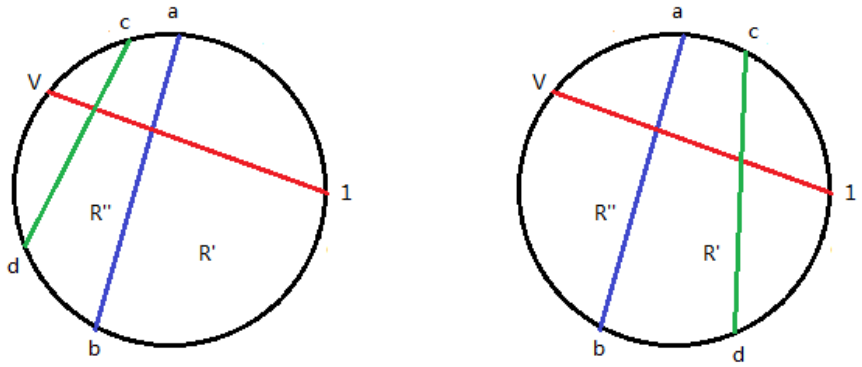
only the fragment separating V from 1, the other half R'' can be neglected after τ. This is also implied in ν_D. Afterwords, in the evolution of R', by using the 3.5 and the fact V is uniform conditionally on R', R' follows the same law of division and lifetime.

In the second case where V is on R'', then two parts have to be taken into considered. On the half R', conditionally V on R'', it is uniform and by the 3.5, $\forall c, d \in R''$

$$[cd] \cap [1V] \Longleftrightarrow [\Phi_{a,b}(c)\Phi_{a,b}(d)] \cap [\Phi_{a,b}(a)\Phi_{a,b}(V)]$$

where $\Phi_{a,b}(a) = 1$ on $\Phi_{a,b}(R'')$ and $\Phi_{a,b}(V)$ conditionally uniform on $\Phi_{a,b}(R')$. Therefore, when it divides, it follows also the same law of ν_D. By the similar argument, $\forall c, d \in R'$

$$[cd] \cap [1V] \Longleftrightarrow [\Psi_{a,b}(c)\Psi_{a,b}(d)] \cap [\Psi_{a,b}(1)\Psi_{a,b}(a)]$$



where $\Psi_{a,b}(1) = 1$ and $\Psi_{a,b}(a)$ is conditionally uniform on $\Psi_{a,b}(R')$, so it follows ν_D in following division.

Finally, we conclude that ν_D is the right dislocation measure which describes the process counting only the fragment separating 1 from V. Considering the fact the lifetime inherits from the process of *figela*, the fragmentation process with parameter (α, ν_D) describes the *figela* process counting only the specified fragments □

We can calculate the Malthusian exponent directly

$$\kappa_{\nu_D}(p) = 1 - \int_0^1 u^{p+2}du - 4\int_{\frac{1}{2}}^1 u(1-u)(u^p + (1-u)^p)du = \frac{p^2+3p-2}{p^2+5p+6}$$

Then $\kappa_{\nu_D}(\beta^*) = 0$ implies that

$$\beta^* = \frac{\sqrt{17}-3}{2}$$

We apply the 3.1 to the fragmentation process $\mathcal{X}_\alpha(t)$

Proposition 3.6. *Let V be a random variable uniformly distributed on $\mathbb{S}_1$ and independent of the figela process $(S_\alpha(t), t \geq 0, \alpha \geq 0)$ and we denote*

$$\mathscr{M}_t^\alpha(x) = \sum_{i=1}^{\infty} m(R_i^{(1,x)}(S_\alpha(t)))^{\beta^*}$$

then we have :
(1) The process $\mathscr{M}_t^\alpha$ is a uniformly integrable martingale and converges almost surely toward a random variable $\mathscr{M}_\infty^V$ which does not depend on $\alpha \geq 0$. Moreover, $\mathscr{M}_\infty^V > 0 a.s.$, and $\mathbb{E}[(\mathscr{M}_\infty^V)^q] < \infty$ for every $q \geq 1$.
(2) There exist a positive random variable $\mathscr{H}_0^V$ such that

$$e^{-t/3}H_{S_0(t)}(1, V) \xrightarrow[t\to\infty]{a.s.} \mathscr{H}_0^V$$

(3) For every $\alpha > 0$, there exists a constant $K_{\nu_D}(\alpha)$ such that

$$t^{-\beta^*/\alpha}H_{S_\alpha(t)}(1, V) \xrightarrow[t\to\infty]{L^2} K_{\nu_D}(\alpha)\mathscr{M}_\infty^V$$

Proof. They are the direct result from 3.1.In (2), the power $-\frac{1}{3}$ comes from $\kappa_{\nu_D}(0)$. □

We take the value $\alpha = 2$ and apply the embedding lemma 2.6

Corollary 3.2.

$$n^{-\beta^*/2} H_n(1, V) \xrightarrow[t\to\infty]{\mathbb{P}} K_{\nu_D}(\alpha)\mathscr{M}_\infty^V$$

This is also correct when we replace V by a fix number x, but it requires more arguments. The reader can refer [1]. Here, we only states the theorem

Theorem 3.2 (Convergence of height function)**.** *Let $x \in \mathbb{S}_1 \setminus \{1\}$*
(1)The process $\mathscr{M}_t^\alpha$ converges almost surely toward a random variable $\mathscr{M}_\infty(x)$ which does not depend on α.
(2)We have $\mathscr{M}_\infty(x) > 0$ a.s and $\mathbb{E}[\mathscr{M}_\infty(x)^q] < \infty$ for every $q \geq 1$. (3)There exist a positive random variable $\mathscr{H}_0$ such that

$$e^{-t/3} H_{S_0(t)}(1, x) \xrightarrow[t\to\infty]{a.s.} \mathscr{H}_0(x)$$

(4)For every $\alpha > 0$, there exists a constant $K_{\nu_D}(\alpha)$ such that

$$t^{-\beta^*/\alpha} H_{S_\alpha(t)}(1, x) \xrightarrow[t\to\infty]{\mathbb{P}} K_{\nu_D}(\alpha)\mathscr{M}_\infty(x)$$

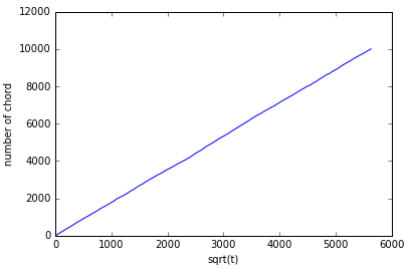
More generally, for every $p \geq 0$, there exists a positive random variable $\mathscr{H}_p(x)$ such that

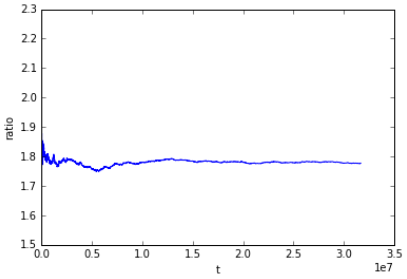
$$e^{t\kappa_{\nu_D}(p)} \sum_{i=0}^{\infty} m(R_i^{1,x}(S_0(t)))^p \xrightarrow[t\to\infty]{a.s} \mathscr{H}_p(x)$$

3.5 Numerical simulation

3.5.1 Simulation of $S_2(t)$

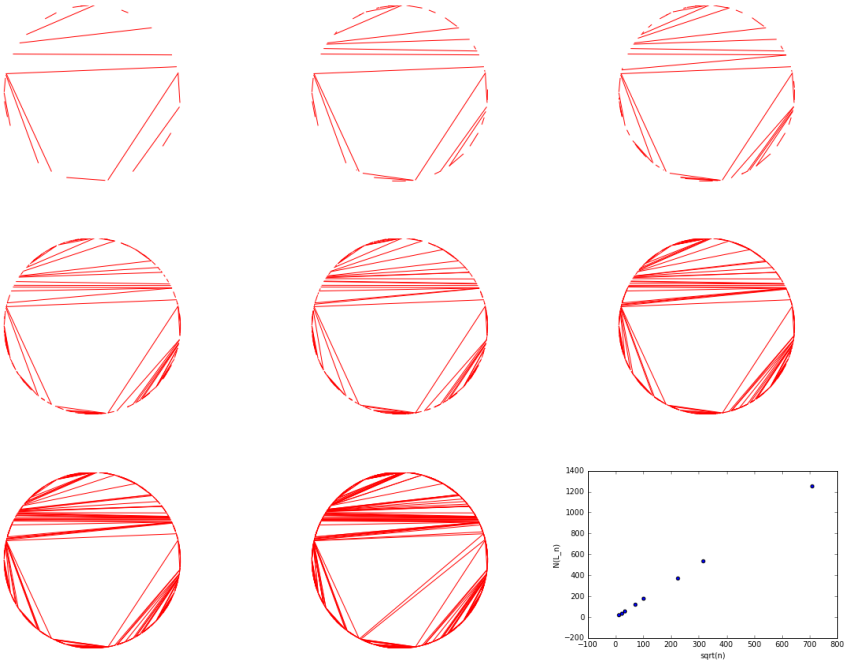
We write a Python program to simulate the evolution of a *figela* process $S_2(t)$. We do 10000 times of division and observe that $\frac{S_2(t)}{\sqrt{t}}$ has really trend to converge. Moreover, we do 100 samples for the case until 100 divisions and obtain the average time $3289 \approx (\frac{100}{\sqrt{\pi}})^2 = 3183$.





3.5.2 Simulation of L_n and Visualisation

We release another simulation of the evolution of L_n. The following is a evolution and we take the time $100, 500, 1000, 5000, 10000, 50000, 100000, 500000$. We show another image which reveals the evolution of $N(L_n)$ and $\sqrt{n}$.



3.5.3 Comments

To add n chords on a disk, the process of L_n needs about $O(n^3)$ complexity, while the fragment process needs only n^2 complexity. So, if the normalized height function can really be used in the numerical simulation, it will accelerate the program. However, the contour function has only definition for continuous function, so how to generalize it to the discrete case, or how do interpolation for building a contour function should be an interesting question.

4 Identifying the limit lamination

Once we get the normalized limit of height function, it is nature to question if we can reconstruct the lamination by the limit function $\mathscr{M}_\infty$? More precisely, we sample ω in the experiment space Ω, then this ω gives a limit of lamination $L_\infty(\omega)$ and also the limit of martingale in 3.2. Using $\mathscr{M}_\infty(\omega)$ We can code a new lamination $L_{g_\infty}(\omega)$ by the method introduced in 2, a natural question is the relation between L_∞ and L_{g_∞}.

The answer is, as we have conjectured, positive. The strategy of proof is to show at first that $L_\infty \subset L_{g_\infty}$, then we demonstrate that in fact L_∞ is maximal, so naturally the two is almost surely equal. The proof of these two parties are contained in the following two subsections, while the third section is conserved for one key lemma of branching random walk in the proof.

4.1 Coding L_∞ by $\mathscr{M}_\infty$

In this subsection, we prove the inclusion relation. At first, we give some detailed analysis about the *figela* process. In the previous subsection, we have stated several theorems where the limit martingale does not depend on the choice of α, here we give an interpretation.

Proposition 4.1 (limit of $S_\alpha(t)$)**.** *(1)We take the increasing limit of $S_\alpha(t)$, it does not depend on the parameter α, that is*

$$S(\infty) = \lim \uparrow S_\alpha(t)$$

(2)L_∞ has two equivalent definitions

$$L_\infty = \overline{\bigcup_{\{x,y\}\in S(\infty)} [xy]}$$

We can also take the closure at before do bijection. That is we define the closure on $\mathbb{S}_1 \times \mathbb{S}_1$ and obtain $S^*(\infty)$. Then

$$L_\infty = \bigcup_{\{x,y\}\in S^*(\infty)} [xy]$$

Moreover, if (x, y) and (x', y') belong to $S^*(\infty)$, the chord $[xy]$ and $[x'y']$ either coincide or do not cross.

Proof. (1)We use the genealogy structure in fragmentation process to couple the *figela* process. We take the notation from 3.2, each fragment in the process is represented by an element u in the binary tree

$$\mathbb{T} = \bigcup_{n\geq 0} \{0, 1\}^n$$

The initial fragment is noted as $\emptyset$ and when it divide, the two are noted respectively R_0, R_1. They have also lifetime: we define a $(e_u)_{u\in\mathbb{T}}$family of independent exponential random variable of parameter 1, then R_0, R_1 are alive respectively until $e_\emptyset + m(R_0)^{-\alpha}e_0, e_\emptyset + m(R_1)^{-\alpha}e_1$. Then, when it divide, the two feet is chosen uniformly on the fragment. By this induction, we can define the *figela* process $S_\alpha(t)$.

The advantage of this construction is that, if we couple the choice of the feet and $(e_u)_{u\in\mathbb{T}}$, for $\alpha' > \alpha > 0$ i.e they have the same sample for $(e_u)_{u\in\mathbb{T}}$ and the sample for the feet in division, the only difference between process is the time of division. We have necessarily

$$S_{\alpha'}(t) \subset S_\alpha(t) \subset S_{\alpha'}(T_{t,\alpha,\alpha'})$$

where $T_{t,\alpha,\alpha'}$ is a random variable almost surely finite. Then we take the increasing limit and we obtain $S(\infty) = S(\infty) = \lim\uparrow S_\alpha(t)$.

(2)We recall that the original definition is

$$L_\infty = \overline{\bigcup_{n\geq 0} L_n}$$

We take $\alpha = 2$ and use the embedding theorem 2.6, then the feet of L_∞ and $S(\infty)$ are the same, so we get a new definition $L_\infty = \overline{\bigcup_{\{x,y\}\in S(\infty)}[xy]}$.

The second definition is exact the result that the bijection from feet pair $\{x, y\}$ to the chord $[xy]$ are continuous and have continuous inverse, so we can change the order of closure and application. □

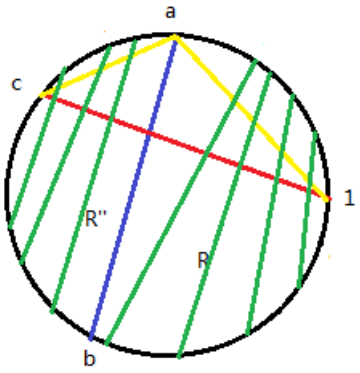
Then we use the limit martingale to obtain the inclusion relation.

Proposition 4.2. *Let $(U_1, U_2), 0 < U_1 < U_2 < 1$, be the first pair of feet, and $a = e^{2\pi i U_1}, b = e^{2\pi i U_2}, M = 1 - (U_2 - U_1)$. Conditionally on the pair U_1, U_2, we have*

$$(\mathscr{M}_\infty(e^{2\pi i(U_1 + r(U_2-U_1))}) - \mathscr{M}_\infty(e^{2\pi i U_1}))_{r\in[0,1]} \overset{(d)}{=} ((1-M)^{\beta^*}\tilde{\mathscr{M}}_\infty(e^{2\pi i r}))_{r\in[0,1]}$$

where $\tilde{\mathscr{M}}_\infty$ is another copy of $\mathscr{M}_\infty$ and independent of M. Moreover, we have

$$\tilde{\mathscr{M}}_\infty(e^{2\pi i r}) > 0$$



Proof. We denote $c = e^{2\pi i(U_1 + r(U_2-U_1))}$ and τ the jump time. Then we claim a simple identity

$$H_{S_\alpha(t+\tau)}(1, c) = 1 + H_{S_\alpha(t+\tau)}(1, a) + H_{S_\alpha(t+\tau)}(a, c)$$

An interpretation of this identity is that, in one sample, the chord who crosses $[1c]$ is $[ab]$ and the one crosses either $[1,a]$ or $[ac]$. This is clear : let $[d_1d_2]$ be a chord different from $[ab]$ in the lamination who crosses $[1c]$. Since it cannot cross $[ab]$, its two feet fall in couple $(\widehat{1a},\widehat{1b})$ or $(\widehat{ca},\widehat{cb})$. The former will cross $[1a]$ and the latter will cross $[ac]$, so we get the identity.

Using the scaling property 3.5 and we get

$$\begin{aligned} t^{-\beta^*/\alpha}H_{S_\alpha(t+\tau)}(1,c) &\overset{(d)}{=} t^{-\beta^*/\alpha} + t^{-\beta^*/\alpha}H_{S_\alpha(t+\tau)}(1,a) \\ &+ (1-M)^{\beta^*}((1-M)^\alpha t)^{-\beta^*/\alpha}H_{S'_\alpha((1-M)^\alpha t)}(1,e^{2\pi ir}) \end{aligned}$$

Conditionally on (U_1,U_2) chosen, we pass to the limit and thanks to the 3.2, we obtain the result. The fact $\tilde{\mathscr{M}}_\infty(e^{2\pi ir})>0$ is just one part of the theorem 3.2. □

Using this result, we obtain immediately the result

Corollary 4.1. *Almost surely, for every $r,s\in[0,1]$ such that $\{e^{2\pi ir},e^{2\pi is}\}\in S(\infty)$, we have $r\overset{g_\infty}{\approx}s$, where $g_\infty(r)=\mathscr{M}_\infty(e^{2\pi ir})$ for $r\in[0,1]$.*

Proof. We prove it by induction. After adding the first chord $[ab]$, in the case where 1 is not the endpoint with a,b, using 4.2 we get $\forall x=e^{2\pi i(U_1+r(U_2-U_1))}$

$$\mathscr{M}_\infty(x)-\mathscr{M}_\infty(a)\overset{(d)}{=}(1-M)^{\beta^*}\tilde{\mathscr{M}}_\infty(e^{2\pi ir})>0 \text{ p.s}$$

We continue the induction and use the 3.5, after adding one chord, the process can be considered as two disks who evolve independently by changing the scale of time, so if we always suppose that 1 in each disk will not be the endpoint of chord, we repeat the 4.2 and obtain the result.

□

Remark. **However, in this argument above, we add always the hypothesis that 1 isn't the endpoint**. We will prove this in the following lemma. This supposition means, in fact, each foot will not be chosen a second time to be the foot of a new chord. Intuitively, it is true since there are only countable point in $S(\infty)$ and every time, a point is chosen with probability 0, so a foot will not be chosen as a foot for the second time. In another word, in the *figela* process, the probability to appear a triangle in its associated lamination is 0.(But it is not the case if we take the closure.)

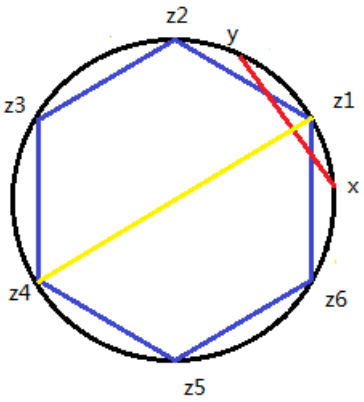
We complete this lemma and finish the proof.

Lemma 4.1 (A specified point with 0 probability to be in the set of feet)**.** *For every $x\in\mathbb{S}_1$,*

$$\mathbb{P}[\forall y\in\mathbb{S}_1\setminus\{x\}:(x,y)\in S^*(\infty)]=0$$

Proof. We give a proof by using the result scaling limit of 3.2. $\forall\epsilon>0$,

$$\begin{aligned} &\mathbb{P}[\exists y\in\mathbb{S}_1,|y-x|>\epsilon,(x,y)\in S^*(\infty)] \\ &=\int \mathbf{1}_{\{\exists y\in\mathbb{S}_1,|y-x|>\epsilon,(x,y)\in S^*(\infty)\}}m(dx) \end{aligned}$$



We will give an estimation for this probability. We take n such that $\frac{1}{n} < \epsilon$ and $z_k = \exp(\frac{2k\pi i}{n})$. Under this condition, it exists at lest one k such that $]z_k z_{-k}[\cap]xy[\neq \emptyset$. Moreover, the event that the two chords cross is equivalent that x is in one fragment that cross $]z_k z_{-k}[$, and this measure almost surely will not increase with the time t, so

$$\begin{aligned} & \int \mathbf{1}_{\{\exists y\in\mathbb{S}_1, |y-x|>\epsilon, (x,y)\in S^*(\infty)\}} m(dx) \\ \leq & \int \mathbf{1}_{\{\exists k, (x,y)\in S^*(\infty),]xy[\cap]z_k z_{-k}[\neq 0\}} m(dx) \\ \leq & \sum_{k=1}^{n}\sum_{i=1} m(R_i^{z_j, z_{-j}(S_0(t))}) \\ \leq & ne^{-t\kappa_{\nu_D}(1)}\mathscr{H}_1(-1) \end{aligned}$$

We calculate

$$\kappa_{\nu_D}(1) = \frac{1^2+3\times 1-2}{1^2+5\times 1+6} = \frac{1}{6}$$

Therefore, the probability is 0 for any $\epsilon > 0$. We take the union and conclude. □

Then, this lemma assures that in the induction of 4.1, each time we can assume that there is no probability that 1 is the endpoint and apply the the argument.

We finish the proof of inclusion relation.

Proposition 4.3 (Inclusion)**.** *Almost surely, we have* $L_\infty \subset L_{g_\infty}$.

Proof. By the 4.1 and 4.1, we know that $\forall\{s,t\} \in S(\infty)$, $s \overset{g_\infty}{\approx} t$, so they have also a chord. This proves

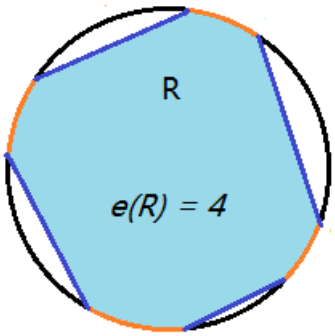
$$L_\infty = \bigcup_{\{x,y\}\in S(\infty)} [xy] \subset L_{g_\infty}$$

Then, we know the L_∞ is the closure of the term in the left hand, while 2.4 shows that the a lamination coded by a continuous function is closed, so we obtain $L_\infty \subset L_{g_\infty}$. □

4.2 Maximality of L_∞

In this subsection, we prove the maximality of L_∞. This will help us finish the proof of the coding the L_∞ by $\mathscr{M}_\infty$. We first introduce another geometry conception, who plays a significant in the proof.

Definition 4.1 (End)**.** Let R be a fragment in a lamination, we define **end** the number of connected component of $R \cap \mathbb{S}_1$ and represent it by $e(R)$.



We take the notation in 4.1 and we note $\mathbb{T}_n$ the *figela* process until the generation n. However, this time, we neglect the information like lifetime, since we have proved that it does not change the geometric property. We will use the information of mass in the following proposition and after we will concentrate only on the evolution of end.

Proposition 4.4 (End as a Markov chain)**.** *Let $(R_u)_{u\in\mathbb{T}}$ be the fragment of a figela process. The evolution of the the end is a Markov chain:*

$$\begin{aligned} e(R_\emptyset) &= 0 \\ \mathbb{P}[e(R_{u0}) = q | e(R_u) = p] &= \frac{1}{1+p}\mathbf{1}_{\{1\le q\le 1+p\}} \end{aligned}$$

Proof. We prove by induction that for every R_u, it has the following property : It has the transition law as described above. Moreover, the mass of each connected component of $R_u \cap \mathbb{S}_1$ is uniformly distributed for $m(R_u)$ given.

The initial state is trivial. We do the induction part. Suppose that R_u has the property above and $e(R_u) = p$, then for R_{u0} what we do is equivalent to add two points a, b uniformly on a circle of length $m(R_u)$, where p points $(c_i)_{1\le i\le p}$ already uniformly distributed on it. Thus, it has $p+2$ points uniformly distributed, which cut the circle into $p+2$ pieces and R_{u0} is the part in counter-clockwise direction from a to b.

Then the probability that R_{u0} has q piece is just the probability that there are q points from a to b. If we count from a and write the order, the uniform

distribution assures that each permutation has the same probability. Therefore

$$\begin{aligned} & \mathbb{P}[e(R_{u0}) = q] \\ = & \frac{\sharp\{\text{Permutations of a,b, } c_i \text{ started by a and b on the (q+2) position}\}}{\sharp\{\text{Permutations of a,b, } c_i \text{ started by a}\}} \\ = & \frac{p!}{(p+1)!} = \frac{1}{p+1} \end{aligned}$$

Moreover, the uniform distribution assures that conditional on $e(R_{u0})$ and $m(R_{u0})$, the mass of each connected component of R_{u0} is also uniformly distributed. □

A **ray** in the genealogy tree is an infinite $\{0,1\}$ sequence and for generation n, we take the first n digits. A ray describes the evolution of the fragmentation process by following always one half in two divisions.

The maximality of L_∞ is the result of the following lemma.

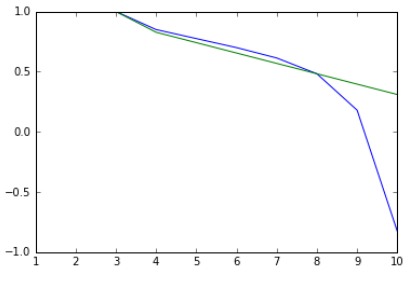
Lemma 4.2 (No ray has always end larger than 3)**.** *Given a figela process and its associated genealogy tree, almost surly, there is no ray along which the the end is always bigger than 3.*

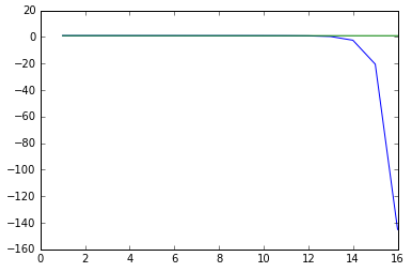
Remark. Generally, it does not suffice to prove the result along a specific ray, since there are 2^n rays in $\mathbb{T}_n$ so in $\mathbb{T}$ there are in fact uncountable rays. This makes the countable additivity powerless.

Remark. Since the Markov chain of $e(R_u)$ has nothing to do with the mass and lifetime, it can be treated as a branching process on the genealogy tree. If we denote p_i the probability that a process starts with end i and no ray of it has always end larger than 3, then by the Markov property, we have a simple recurrence relation

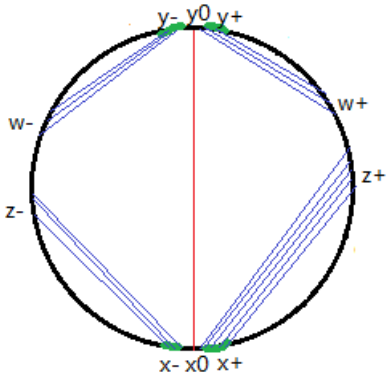
$$\begin{cases} p_1 = p_2 = p_3 = 1 \\ p_k = \frac{1}{k+1} \sum_{i=1}^{k+1} p_i p_{k+2-i}, \text{ if } k \geq 4 \end{cases}$$

A solution for this series should satisfy that $p_k \in [0,1], \forall k$. The numerical result shows that if $p_4 \neq 1$, it will have progressive fall to negative in value in cause of the non-linearity. The following image shows result that we take $p_4 = 0.15, 0.001$.





The proof of this lemma is very technical and contains a lot of quantitative analysis, so we keep this part in the next subsection. Here, we use this lemma to prove the maximality.



Proposition 4.5. *Almost surely, L_∞ is a maximal lamination of $\overline{\mathbb{D}}$.*

Proof. We first prove that the feet set of $S(\infty)$ is dense on $\overline{\mathbb{D}}$. Suppose that x_0, y_0 are two points on $\mathbb{S}_1$ and they are neither feet nor the limit of the feet, so $\exists \delta > 0$ such that in the δ neighbourhood of them, there are no feet. To study the probability of this case, we just take a L_n process and each time, the probability that no points in the δ neighbourhood is less than $1 - 4\delta^2$. The sum finite of geometric series and Borel-Cantelli lemma implies that the probability is 0.

Secondly, we prove the maximality. Suppose that we do a sample ω and $L_\infty(\omega)$ is not maximal. then we can add one chord $[x_0 y_0]$ without crossing any other existed chord. However, the density assure that

$$\begin{aligned} \exists (x_n^+)_{n\geq 0}, (x_n^-)_{n\geq 0}, (y_n^+)_{n\geq 0}, (y_n^-)_{n\geq 0} \in \mathbb{S}_1 \\ x_0 = \lim_{n\to\infty} x_n^+ = \lim_{n\to\infty} x_n^- \\ y_0 = \lim_{n\to\infty} y_n^+ = \lim_{n\to\infty} y_n^- \end{aligned}$$

The two series $(x_n^+)_{n\geq 0}, (x_n^-)_{n\geq 0}$ approximate x_0 from two sides. We design a function $f(x_n^+)$ the other foot which forms a chord with x_n^+. Then, the compactness of $\mathbb{S}_1$ induces that $\{f(x_n^+)\}_{n\geq 0}$ has a limit value different from y_0, otherwise the pair $\{x_0, y_0\}$ is in the closure of $S(\infty)$ so in L_∞. Additionally, the limit value should be on the right of $[x_0 y_0]$ in order to avoid crossing. Thus,

$$\exists z^+ \in \mathbb{S}_1, \exists (x_{n_k}^+)_{k\geq 0} \text{ subseries of } (x_n^+)_{n\geq 0} \text{ s.t. } z^+ = \lim_{k\to\infty} x_{n_k}^+$$

In fact, we can get some stronger result since the no crossing argument assures that f is monotone in one direction on $\mathbb{S}_1$. Then,

$$z^+ = \lim_{n\to\infty} x_n^+$$

The same argument is also correct for $(y_n^+)_{n\geq 0}$ and we denote that $w^+ = \lim_{n\to\infty} y_n^+$. On the left of $[x_0 y_0]$ there are also limits $z-, w-$. Although w^+, z^+ can coincide, so do w^-, z^-, the fragment containing these points have always at least 4 connected component on $\mathbb{S}_1$.

Thanks to 4.2, we know the sample ω has 0 probability, so almost surely L_∞ is a maximal lamination. □

Proof. **proof of 1.3** By 4.3 and 4.5, we deduce that almost surly, $L_\infty = L_{g_\infty}$. Then, by 2.4 we know the relation $\overset{g_\infty}{\approx}$ coincides with the relation $\overset{g_\infty}{\simeq}$, so we draw a chord simply when

$$\mathscr{M}_\infty(\omega)(x) = \mathscr{M}_\infty(\omega)(y) = \min_{z \in Arc(x,y)} \mathscr{M}_\infty(\omega)(z)$$

□

4.3 Proof of the key lemma

Reader can find the proof in [1] and [3] contains more details about this branching random walk.

A An analytic proof of a key lemma in random lamination

[1] studies the limit of a random recursive lamination and one of the key lemma is the its maximality. The authors propose to study a more analytic proof of this lemma, which reduces to a recursive series. In this report, we prove it.

A.1 Introduction

In [1], the authors prove that we can code the limit of random recursive lamination by its associated fragmentation martingale limit and one key lemma is the maximality of the limiting laminations(proposition 5.4). This lemma depends strongly on the following proposition.

Proposition A.1. *In the genealogical tree of fragments, almost surely, there is no ray along which all fragments have eventually strictly more than 3 ends.*

Reader can check the article for the definition and notation. In fact, along one ray, the number of end evolves as a Markov chain

$$p_2(x,y) = \frac{1}{x+1}\mathbf{1}_{1 \le y \le x+1}$$

, while the other child of the fragment has $(x+2-y)$ ends. In [1], the authors take analyse on the branch process and prove the above proposition. In [3], a more fine estimation of the probability is given for the case that along a branch until generation n, all the number of ends are bigger than 4.

In fact, if we starts a fragment process with k ends and denote p_k the probability that there exits no infinite ray starting from $\emptyset$ along which all the number of end are strictly greater than 3. Then p_i satisfy a simple recursive relation

$$\begin{cases} p_1 = p_2 = p_3 = 1 \\ p_n = \frac{1}{n+1}(p_1 p_{n+1} + p_2 p_n + \dots p_n p_2 + p_{n+1} p_1) \end{cases}$$

When p_4 is given, this recursive system is determined. So, we conjecture that the only solution that makes $p_n \in [0,1], \forall n$ should be that $p_4 = 1$. The numerical test also verifies our conjecture. This recursive system is also remarked in the

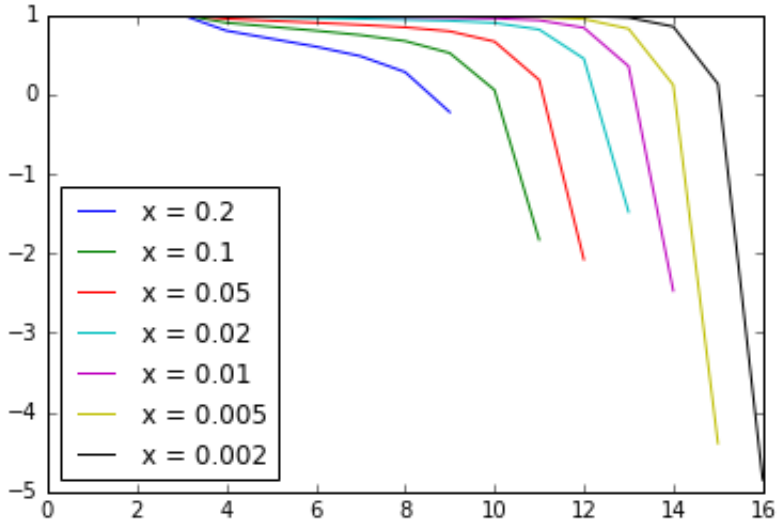


Figure 1: Evolution of series for $p_4 = 1 - x$

article. If we have proved the uniqueness of this system, it will give another more analytic proof for the maximality of limiting lamination.

Meanwhile, we say something about the numerical experiences. In fact, it is easy to give some numerical experiences for an initial condition $p_4 = 1 - x, x \in (0, 1)$. The numerical results show that the series will, finally, fall down below 0, no matter how small x is. Moreover, the series decrease at first slowly, but drastically later. All these inspire us to study the property of this interesting non-linear system.

Although this recursive relation is simple, it requires also detailed analysis due to its non-linearity. In the rest of this report, we will study it. We organize the report as following : In section 2, we will give an polynomial representation of this series and its properties, which help us prove the uniqueness of the positive solution. In section 3, we will give some further analysis for the evolution of the system and we will see that p_n decreases with a speed of high order polynomial rather than linearly when n is big. Moreover, we will use this estimation to explain the "cut-off" phenomena. In the last section, readers can find the numerical experiences which illustrate our theorems.

A.2 Uniqueness of the positive solution

The above system can be written as

$$p_{n+1} = p_n + (p_n - p_{n-1}) + \frac{1}{2}\sum_{i=4}^{n-2}(p_n - p_i p_{n+2-i}) \tag{3}$$

This is an important identity and we will use it many times later. From this identity, we know that if we let $p_4 = 1 - x, x \in [0, 1]$, then every p_n can be written as a polynomial of x. Moreover, these polynomials have some good properties and will help us obtain the uniqueness of the system.

Theorem A.1. *p_4 is defined as above, then $\forall n \geq 4$, p_n is a polynomial of x in the form*

$$p_n = 1 - \frac{n-2}{2}x - x^2 A_n(x) \tag{4}$$

where $A_n(x)$ is also a polynomial of x. We write it as A_n for short. $\{A_n\}_{n\geq 4}$ satisfy that

- ***(Positivity)*** $A_n \geq 0, \forall n \geq 4$. *More precisely, it is strict when $n \geq 7$.*
- ***(Convexity 1)*** $A_{n+1} - A_n \geq A_n - A_{n-1}, \forall n \geq 5$. *More precisely, it is strict when $n \geq 6$.*
- ***(Convexity 2)*** $A_n \geq A_i + A_{n+2-i}, \forall i \geq 4, n \geq 6$. *More precisely, it is strict when $n \geq 7$.*

Here we define $A \geq B$ if all the coefficients of $A - B$ is positive. We define $A > B$ if $A \geq B$ and $A - B \neq 0$.

Proof. We can calculate the expression until p_8. That is

$$\begin{aligned} p_1 = p_2 = p_3 &= 1 \\ p_4 = 1 - x &, \quad A_4 = 0 \\ p_5 = 1 - \frac{3}{2}x &, \quad A_5 = 0 \\ p_6 = 1 - 2x &, \quad A_6 = 0 \\ p_7 = 1 - \frac{5}{2}x - \frac{1}{2}x^2 &, \quad A_7 = \frac{1}{2} \\ p_8 = 1 - 3x - 3x^2 &, \quad A_8 = 3 \end{aligned}$$

So it is easy to check that they satisfy the properties and we start the proof by recurrence from $n + 1 = 9$.

Suppose that the polynomial expression and all the properties are correct for $p_k, 4 \leq k \leq n$, then for p_{n+1} we plunge the polynomial in (3)

$$\begin{aligned} p_{n+1} &= p_n + (p_n - p_{n-1}) + \frac{1}{2}\sum_{i=4}^{n-2}(p_n - p_i p_{n+2-i}) \\ &= (1 - \frac{n-2}{2}x - x^2 A_n) + [(1 - \frac{n-2}{2}x - x^2 A_n) - (1 - \frac{n-3}{2}x - x^2 A_{n-1})] + \frac{1}{2}\sum_{i=4}^{n-2}(p_n - p_i p_{n+2-i}) \\ &= 1 - \frac{n-1}{2}x - x^2[A_n + (A_n - A_{n-1})] + \frac{1}{2}\sum_{i=4}^{n-2}(p_n - p_i p_{n+2-i}) \end{aligned}$$

We develop the last term, in fact we claim $\forall 4 \leq i \leq n-2, n \geq 6$,

$$p_n - p_i p_{n+2-i} = -x^2 B_{n,i} \tag{5}$$

where $B_{n,i}$ is a polynomial of x and $B_{n,i} > 0$.

This claim can be checked directly by calculus

$$\begin{aligned} & p_n - p_i p_{n+2-i} \\ = \; & (1 - \frac{n-2}{2}x - x^2 A_n) - (1 - \frac{i-2}{2}x - x^2 A_i)(1 - \frac{n-i}{2}x - x^2 A_{n+2-i}) \\ = \; & (1 - \frac{n-2}{2}x - x^2 A_n) - (1 - \frac{n-2}{2}x + \frac{(i-2)(n-i)}{4}x^2 - x^2(A_{n+2-i} + A_i) \\ & + x^3(\frac{n-i}{2}A_i + \frac{i-2}{2}A_{n+2-i}) + x^4 A_i A_{n+2-i}) \\ = \; & -x^2[(A_n - A_{n+2-i} - A_i) + \frac{(i-2)(n-i)}{4} + x\frac{n-i}{2}A_i + x\frac{i-2}{2}A_{n+2-i} + x^2 A_i A_{n+2-i}] \end{aligned}$$

Compared with (5) we get

$$B_{n,i} = (A_n - A_{n+2-i} - A_i) + \frac{(i-2)(n-i)}{4} + x\frac{n-i}{2}A_i + x\frac{i-2}{2}A_{n+2-i} + x^2 A_i A_{n+2-i} \tag{6}$$

The convexity 2 assures that $B_{n,i} > 0$.

Then we simplify the expression

$$p_{n+1} = 1 - \frac{n-1}{2}x - x^2[A_n + (A_n - A_{n-1}) + \frac{1}{2}\sum_{i=4}^{n-2} B_{n,i}] \tag{7}$$

We compare this equation with the standard expression and get

$$A_{n+1} = A_n + (A_n - A_{n-1}) + \frac{1}{2}\sum_{i=4}^{n-2} B_{n,i} \tag{8}$$

This equation permit us to do recurrence.

Positivity: Since $A_n - A_{n-1} \geq A_8 - A_7 > 0$ and $A_n > 0, B_{n,i} > 0$, we obtain that A_{n+1} is also strictly positive.

Convexity 1: We subtract this item by A_n.

$$A_{n+1} - A_n = (A_n - A_{n-1}) + \frac{1}{2}\sum_{i=4}^{n-2} B_{n,i} > A_n - A_{n-1}$$

Convexity 2: We use the Convexity 2 of A_n, that is $\forall 4 \leq i \leq n-2$

$$\begin{aligned} A_{n+1} &= A_n + (A_n - A_{n-1}) + \frac{1}{2}\sum_{i=4}^{n-2} B_{n,i} \\ &> A_i + A_{n+2-i} + (A_n - A_{n-1}) + \frac{1}{2}\sum_{i=4}^{n-2} B_{n,i} \end{aligned}$$

The Convexity 1 induces that

$$A_n - A_{n-1} > A_{n-1} - A_{n-2} \geq A_{n+3-i} - A_{n+2-i}$$

, therefore

$$\begin{aligned} A_{n+1} &> A_i + A_{n+2-i} + (A_{n+3-i} - A_{n+2-i}) + \frac{1}{2}\sum_{i=4}^{n-2} B_{n,i} \\ &= A_i + A_{n+3-i} + \frac{1}{2}\sum_{i=4}^{n-2} B_{n,i} \\ &> A_i + A_{n+3-i} \end{aligned}$$

The case $i = n-1$ is same as the case $i = 4$, so the convexity 2 is also correct for $n+1$ and we finish the recurrence. □

These properties imply directly the uniqueness of the positive solution.

Corollary A.1. *The unique solution positive of the system is that $p_n = 1$.*

Proof.

$$\begin{aligned} 0 \leq p_n &= 1 - \frac{n-2}{2}x - x^2 A_n \leq 1 - \frac{n-2}{2}x \\ &\Rightarrow 0 \leq x \leq \frac{2}{n-2}, \forall n \\ &\Rightarrow x = 0 \end{aligned}$$

□

This concludes the uniqueness of the system and it gives another proof for the maximality of random recursive lamination defined in [1].

A.3 Speed of the decrement

Using the result in last section that

$$p_n = 1 - \frac{n-2}{2}x - x^2 A_n \leq 1 - \frac{n-2}{2}x$$

we can also prove that the life time of positive solution has upper bound $\lfloor \frac{2}{x} \rfloor + 2$. If we just look at the first order, this series decrease linearly. However, we see in the numerical experience that the value falls in fact quickly when n is big and has the phenomena of "cut-off". This is due to the fact that we treat A_n as just a number positive. In this section, we give a more precise analysis on A_n and improve our result.

Theorem A.2. ***(Convexity 2+)*** *We can give a better second order difference estimation for A_n*

$$D^2(A_n) = A_{n+1} - 2A_n + A_{n-1} = \Omega(n^3) \tag{9}$$

This also improves the estimation of A_n $\forall x \geq 0, A_n = \Omega(n^5)$. So there exists $a > 0$ and for n big enough

$$p_n \leq 1 - \frac{n-2}{2}x - an^5 x^2$$

Proof. We use the recursive relation of A_n in (8)

$$A_{n+1} = A_n + (A_n - A_{n-1}) + \frac{1}{2}\sum_{i=4}^{n-2} B_{n,i}$$

so that

$$D^2(A_n) = \frac{1}{2}\sum_{i=4}^{n-2} B_{n,i} \tag{10}$$

where $B_{n,i}$ is defined in (5) and has an expression in (6). To obtain a better lower bound of $B_{n,i}$, we keep the constant term

$$\begin{aligned} B_{n,i} &= (A_n - A_{n+2-i} - A_i) + \frac{(i-2)(n-i)}{4} + x\frac{n-i}{2}A_i + x\frac{i-2}{2}A_{n+2-i} + x^2 A_i A_{n+2-i} \\ &\geq \frac{(i-2)(n-i)}{4} \end{aligned}$$

Then we plunge it to (10)

$$\begin{aligned} D^2(A_n) &\geq \frac{1}{2}\sum_{i=4}^{n-2}\frac{(i-2)(n-i)}{4} \\ &= \frac{1}{2}[\sum_{i=4}^{n-2}\frac{i-2}{4}n - \sum_{i=4}^{n-2}\frac{i^2}{4} + \sum_{i=4}^{n-2}\frac{2i}{4}] \\ &\geq \frac{1-\epsilon}{48}n^3 \end{aligned}$$

$\forall \epsilon \in (0,1)$ and n big enough. We iterate the second order difference of A_n and get for n big enough

$$A_n \geq \frac{1-\epsilon}{960}n^5$$

This implies a better estimation of p_n by plunge the A_n in equation (4)

$$p_n \leq 1 - \frac{n-2}{2}x - \frac{1-\epsilon}{960}n^5x^2 \tag{11}$$

□

.

So we give a better estimation of the decrease speed, but it isn't the real speed. As we know, if we repeat the argument above, we have hope to continue improving the precision.

Remark. An intuitive (but false!) explanation for "cut-off" phenomena is that when x is small, the second term in (11) can be neglected when n is also small. However, power of 5 increases fast so when n comes to $\lfloor(\frac{1000}{x})^{\frac{1}{5}}\rfloor$, it becomes an counterpart of the first term. After this point, the second term plays the major role so we see a fast decrease.

This argument has some parts positive : in numerical experiences, the "cut-off" starts from a critical point. However, the polynomial decreases cannot catch up the real speed in numerical experiences, which finish the fall down just in 2 or 3 steps. A more precise argument is proposed in the next section.

A.4 Lifetime and critical point

Although we have known a lot about the recursive system, why it appears a sudden fall in experience still interests us. In this section, we explain it mathematically. We will see that in fact, the system has a critical point, from which this series starts a decrease of exponential speed.

We define the **lifetime** of the system as

$$N_+ = \max\{n | \forall 1 \leq i \leq n, p_i \geq 0\}$$

The result in last section gives a large upper bound of the lifetime at least.

Corollary A.2. $N_+ \leq \lfloor (\frac{1000}{x^2})^{\frac{1}{5}} \rfloor \wedge (\lfloor \frac{2}{x} \rfloor + 2)$

Proof. The two estimation come directly from

$$0 \leq p_n \leq 1 - \frac{n-2}{2}x$$
$$0 \leq p_n \leq 1 - \frac{1-\epsilon}{960}n^5x^2$$

□

The real reason for "cut-off" comes from the following theorem.

Theorem A.3. ***(Cut-off)*** *We define a* ***critical point*** *of the system as*

$$n_c = \inf\{n | p_n \leq 1 - (n-2)x\} \tag{12}$$

Then from n_c, p_n decreases exponentially, that is $\forall k \geq 1$,

$$p_{n_c+k} \leq 1 - \left(\frac{n_c + k - 2}{2}\right)x - C\left(\frac{n_c}{2}\right)^{k-1}x \tag{13}$$

Before proving this theorem, we would like to know why this describes "cut-off". In fact, combing (11), this theorem means

$$\forall 4 \leq n < n_c, 1 - (n-2)x < p_n < 1 - \frac{n-2}{2}x$$

so they decease almost linearly. But no matter how slow this linear speed is, after n_c, p_n falls exponentially. That is the vision effect of "cut-off" seen in the experience.

We prove the theorem above.

Proof. Firstly, the definition of critical point is well defined since $D^2(A_n) > 0$ implies that $\frac{A_n}{n}$ is monotone increasing, so the critical point is well-defined. Using (4) and the definition of n_c

$$\begin{aligned} 1 - \frac{n-2}{2}x - x^2 A_{n_c} &\leq 1 - (n-2)x \\ \Leftrightarrow xA_{n_c} &\geq \frac{n_c - 2}{2} \end{aligned}$$

At the same time,

$$\forall n < n_c, xA_n < \frac{n-2}{2}$$

.

Using equation (8) and equation (6)

$$\begin{aligned} A_{n+1} &= [A_n + (A_n - A_{n_c-1}) + \frac{1}{2}\sum_{i=4}^{n-2} B_{n,i}] \\ &\geq \frac{n-1}{2}A_{n_c} - \sum_{i=4}^{n-1} A_i + \sum_{i=4}^{n_c-2} \frac{(i-2)(n-i)}{4} \\ &\geq \frac{n-1}{2}A_n - \sum_{i=4}^{n-1} A_i + \frac{1}{100}n^3 \end{aligned}$$

Therefore, we obtain

$$A_{n+1} \geq \frac{n-1}{2}A_n - \sum_{i=4}^{n-1} A_i + \frac{1}{100}n^3 \tag{14}$$

and we will use this inequality to get our result.

We claim a stronger convexity for A_n :

Proposition A.2. ***(Convexity 3)*** $\exists C > 0, \forall k \geq 1$

$$x^2 A_{n_c+k} \geq C\left(\frac{n_c}{2}\right)^{k-1}$$

This can be deduced by (14) after careful calculus. The first several terms should be done manually. Using the upper bound of A_n for $n < n_c$ and lower bound of A_{n_c} and the fact that $A_4 = A_5 = A_6 = 0$, we have

$$\begin{aligned} x^2 A_{n_c+1} &\geq x^2\left[\frac{n_c-1}{2}A_{n_c} - \sum_{i=7}^{n_c-1} A_i + \frac{1}{100}n^3\right] \\ &> x\left[\left(\frac{n_c-1}{2}\right)\left(\frac{n_c-2}{2}\right) - \left(\frac{n_c-7}{2}\right)\left(\frac{n_c-3}{2} + \frac{7-2}{2}\right)\right] \\ &= \left(\frac{n_c+8}{2}\right)x \end{aligned}$$

.

We continue the calculus for A_{n_c+2}, A_{n_c+3}, we will see that the increment accumulate thanks to the first term A_{n_c+1} achieves just a very small progress - no matter how small it is, it will become enormous later. That is why we should do the calculus very carefully here.

$$\begin{aligned}
x^2 A_{n_c+2} &\geq x^2\left[\frac{n_c}{2}A_{n_c+1} - \sum_{i=7}^{n_c-1} A_i - A_{n_c} + \frac{1}{100}n^3\right] \\
&\geq x^2\left(\frac{n_c-2}{2}A_{n_c+1} - \sum_{i=7}^{n_c-1} A_i\right) + x^2\left(A_{n_c+1} - A_{n_c}\right) \\
&\geq x\left[\left(\frac{n_c+8}{2}\right)\left(\frac{n_c-2}{2}\right) - \left(\frac{n_c-7}{2}\right)\left(\frac{n_c-3}{2} + \frac{7-2}{2}\right)\right] \\
&= \frac{11n_c-2}{4}x \\
&\geq \frac{5}{2}n_c x
\end{aligned}$$

$$\begin{aligned}
x^2 A_{n_c+3} &\geq x^2\left[\frac{n_c+1}{2}A_{n_c+2} - \sum_{i=7}^{n_c-1} A_i - A_{n_c} - A_{n_c+1} + \frac{1}{100}n^3\right] \\
&\geq x^2\left(\frac{n_c-3}{2}A_{n_c+2} - \sum_{i=7}^{n_c-1} A_i\right) + x^2\left(A_{n_c+2} - A_{n_c+1}\right) + x^2\left(A_{n_c+2} - A_{n_c}\right) \\
&\geq x\left[\frac{5n_c(n_c-3)}{4} - \frac{(n_c-7)(n_c+2)}{4}\right] \\
&= \frac{4n_c^2-10n_c+14}{4}x \\
&\geq \frac{1}{2}n_c^2 x
\end{aligned}$$

From A_{n_c+4}, the calculus become easier since A_{n_c+3} itself compensate the sum $\sum_{i=7}^{n_c-1} A_i < \frac{(n_c-7)(n_c+2)}{4}$.

$$\begin{aligned}
x^2 A_{n_c+4} &\geq x^2\left(\frac{n_c+2}{2}A_{n_c+3} - \sum_{i=7}^{n_c-1} A_i - A_{n_c} - A_{n_c+1} - A_{n_c+2}\right) \\
&\geq x^2\left(\frac{n_c-6}{2}\right)A_{n_c+3}
\end{aligned}$$

We assume $n_c \geq 10$ since that is the case non-trivial. Then $A_{n_c+4} \geq 2A_{n_c+3}$.
We do recurrence, that is

$$\begin{aligned}
x^2 A_{n_c+k+1} &\geq x^2\left(\frac{n_c+k-1}{2}A_{n_c+k} - \sum_{i=7}^{n_c-1} A_i - \sum_{j=0}^{k-1} A_{n_c+j}\right) \\
&\geq x^2\left(\frac{n_c+k-7}{2}\right)A_{n_c+k} + x^2(A_{n_c+k} - \sum_{i=7}^{n_c-1} A_i) + x^2(A_{n_c+k} - A_{n_c} - A_{n_c+1}) \\
&\quad + x^2(A_{n_c+k} - \sum_{j=2}^{k-1} A_{n_c+j}) \\
&\geq x^2\left(\frac{n_c+k-7}{2}\right)A_{n_c+k}
\end{aligned}$$

While k becomes bigger than 7, the estimation becomes finally $A_{n_c+k+1} \geq \frac{n_c}{2} A_{n_c+k}$. Therefore, we conclude that A_n has an exponential increment. We put this result into the expression of p_n in equation(4) and we finish the proof. □

Remark. In our proof, we neglects totally the term $\frac{1}{100}n^3$. In fact, it contributes also the exponentially decrement.

A.5 Numerical experiences

We use python to calculate the first 16 polynomials of A_n, whose coefficients are all positive.

$$\begin{aligned}
A_4 &= A_5 = A_6 = 0 \\
A_7 &= 0.5 \\
A_8 &= 3.0 \\
A_9 &= 13.125 \\
A_{10} &= 54.5 + 0.5x \\
A_{11} &= 237.375 + 6.0x \\
A_{12} &= 1128.75 + 48.125x \\
A_{13} &= 5918.0 + 339.625x + 0.625x^2 \\
A_{14} &= 34095.75 + 2337.5x + 12.0x^2 \\
A_{15} &= 214300.125 + 16436.875x + 146.5625x^2 \\
A_{16} &= 1458692.375 + 120600.375x + 1505.0x^2 + 0.875x^3
\end{aligned}$$

Moreover, if we implement the second order difference

$$D^2(A_n) = A_{n+1} - 2A_n + A_{n-1}$$

, we find that their coefficients are also positive.

$$\begin{aligned}
D^2(A_5) &= 0 \\
D^2(A_6) &= 0.5 \\
D^2(A_7) &= 2.0 \\
D^2(A_8) &= 7.625 \\
D^2(A_9) &= 31.25 + 0.5x \\
D^2(A_{10}) &= 141.5 + 5.0x \\
D^2(A_{11}) &= 708.5 + 36.625x \\
D^2(A_{12}) &= 3897.875 + 249.375x + 0.625x^2 \\
D^2(A_{13}) &= 23388.5 + 1706.375x + 10.75x^2 \\
D^2(A_{14}) &= 152026.625 + 12101.5x + 123.1875x^2 \\
D^2(A_{15}) &= 1064187.875 + 90064.125x + 1223.875x^2 + 0.875x^3
\end{aligned}$$

The following two images shows the speed of decrement. We see that as we have predicted, once the series touch the curve $1-(n-2)x$, it accelerates to decrease and it's faster than $1 - \frac{(n-2)}{2}x - \frac{1}{1000}x^5$.

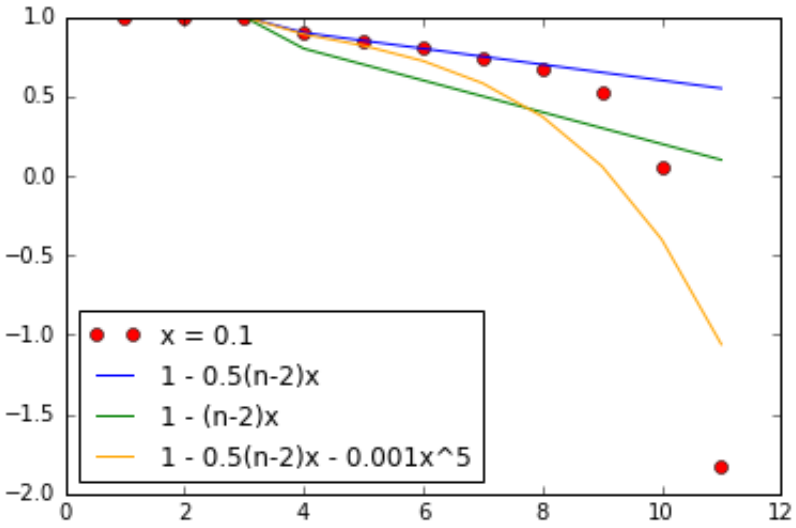


Figure 2: Comparison with different approximation for $x = 0.1$

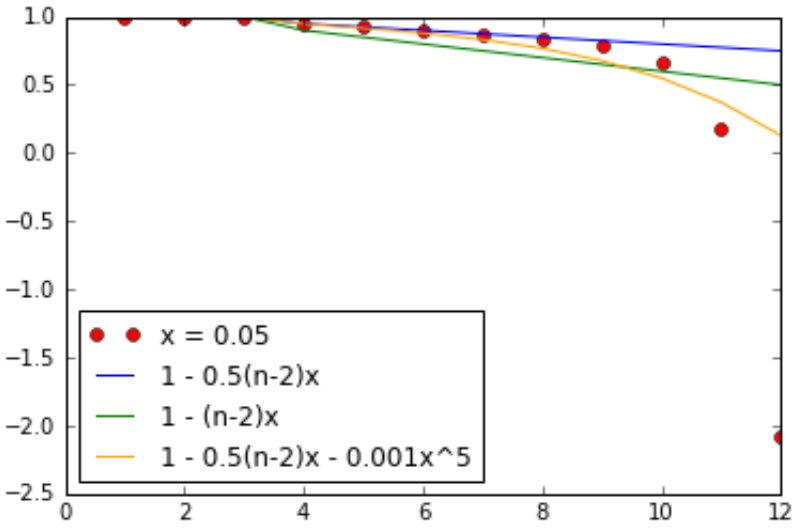


Figure 3: Comparison with different approximation for $x = 0.05$

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