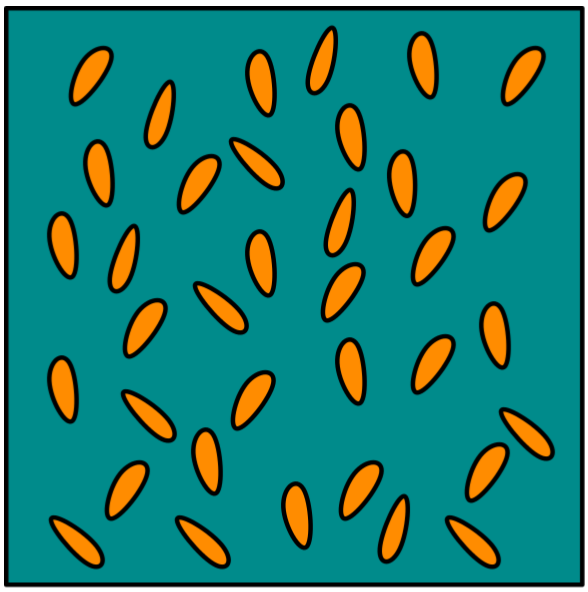
Mémoire de M2

Estimation uniforme sur une méthode itérative pour le problème de homogénéisation

Chenlin GU Ecole Polytechnique X2014



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Chapter 1

Preliminary

1.1 Main story : homogenization

Dirichlet problem is a very classical partial differential equation, which describes, for example, the potential in a domain $U \subset \mathbb{R}^d$ with boundary condition g and with the source f. It has a form like

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{1.1.1}$$

We call it also Poisson equation and in the case where $f = 0$, we call it Laplace equation. This is a very classical problem in maths and has been well studied for long time. However, when we deduce this equation, sometimes we simplify a little the condition in reality. For example, the conductance here is constant, which corresponds to the case that the media is very steady in all the point in U. This is the case in some model but not the case in other situations. For example, if there are two types of media, sometimes we use another equation like divergence form

$$\begin{cases} -\nabla \cdot \mathbf{a}\nabla u = f & \text{in } U, \\ u = g & \text{on } \partial U \end{cases} \tag{1.1.2}$$

to describe the model and we allow that the conductance matrix $\mathbf{a} : U \to \mathbb{R}^{d\times d}$ to take different values depending on the point. The story of homogenization starts from these two equations : eq. (1.1.1) is easier to study in maths and has very good regularity, but it is too simplified and loses some details information. eq. (1.1.2) is little more complicated, while it contains all the information of the media. However, to record all the information of the conductance matrix, it will take many memory and the modeling is also very expensive. Some question is very natural : in which situation we should use eq. (1.1.1) for modeling and which situation we should use eq. (1.1.2) ?

This question should be formulated more precisely since how to model a concrete problem should be sometimes studied case by case. But if we set a conductance $\mathbf{a}^\varepsilon$ that has some spatial structure like $\mathbf{a}^\varepsilon$ is ε-periodic or $\mathbf{a}^\varepsilon$ is sampled i.i.d in small area of length ε, very intuitively, we believe that when $\varepsilon \ll diam(U)$, we may replace the conductance by an *effective conductance* $\bar{\mathbf{a}}$.

This is the main idea of the theory of homogenization, we state it more formally. Suppose that $\mathbf{a}$ is 1-periodic or stationary with after translation of integer, and $\mathbf{a}^\varepsilon = \mathbf{a}(\frac{\cdot}{\varepsilon})$. We study the solution of the equation

$$\begin{cases} -\nabla \cdot \mathbf{a}^\varepsilon \nabla u^\varepsilon = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{1.1.3}$$

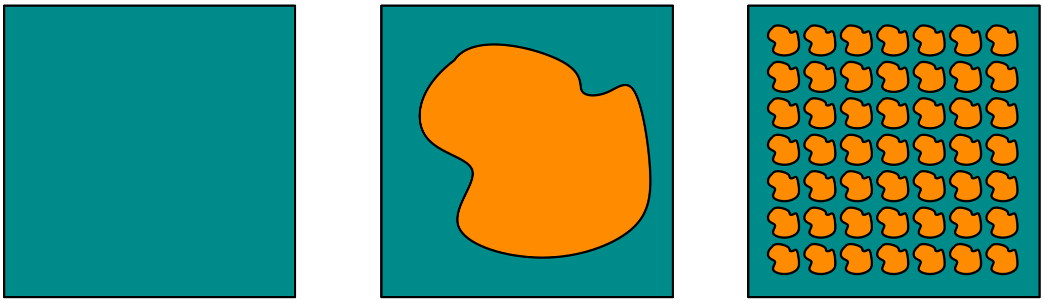

Figure 1.1: The pictures from left to right describe a homogeneous media, an environment of two types of media and a media of periodic structure. In the first case, the equation is simple, while in the third case, we hope to profit of the periodic structure to reduce the complexity of calculus or storage of memory.

This equation is called *cell problem* or *heterogeneous equation* in homogenization theory. It appears especially in the modeling of molecular problem, equation in disordered media/ on random conductance/ on porous area, problem about the design of periodic structure etc. Here we focus on divergence form and many questions can be asked.

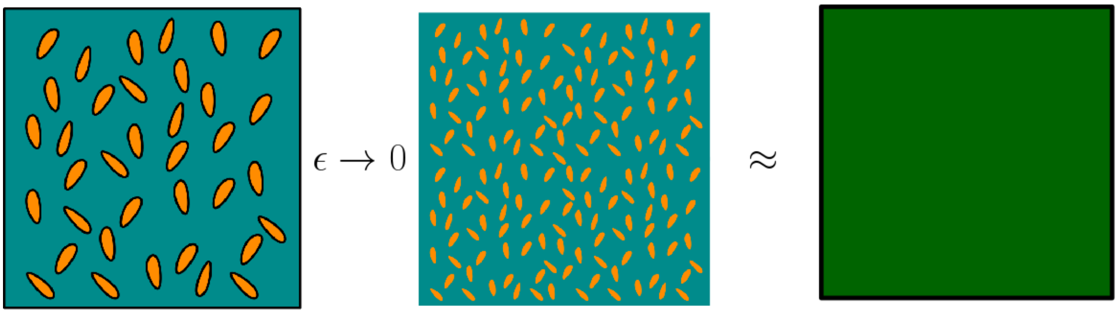

Figure 1.2: When we take ε goes to 0, we believe that the conductance of disordered media will be averaged to an effective conductance $\bar{\mathbf{a}}$.

1. When $\varepsilon \ll diam(U)$, can we find an effective conductance matrix $\bar{\mathbf{a}}$ constant for all $x \in U$ such that the solution $\bar{u}$ of
$$\begin{cases} -\nabla \cdot \bar{\mathbf{a}} \nabla \bar{u} = f & \text{in } U, \\ u = g & \text{on } \partial U \end{cases} \tag{1.1.4}$$
is a limit when ε goes to 0 ? And in which sense with which speed ? This question is of special interest of computational mathematics since to solve an equation of constant conductance matrix is easier than that of a general conductance matrix.

2. Can we find a reverse engineering ? That is to say approximate the solution u^ε by that of $\bar{u}$. Of course, when ε is small, $\bar{u}$ itself is an approximate, but can we pay some more cost (but still less than to solve direct eq. (1.1.3)) to make the approximation more precise than just $\bar{u}$?

3. What is the regularity of the solution u^ε when $f = 0$. We know the regularity of Laplace equation is very good, but does the periodicity or stationarity also helps improve the regularity ?

4. Due to the connection between Dirichlet problem and diffusion process, what is the interpretation of homogenization in the associated diffusion process ?

In this report, we try to review the answer of these questions. In Chapter 1 we collect some basic knowledge about analysis and elliptic equation. In Chapter 2, we review the theory of abstract qualitative theory of homogenization. It says sometimes the existence of the limit using the weak compactness argument or ergodic theory argument, so they are a little abstract and the speed is not explicit enough for numerical analysis. In Chapter 3, we introduce some recent progress in this area, so that we know some explicit speed of homogenization. Chapter 4 is one final project about the bound of one iterative algorithm studied during the internship.

1.2 Functional analysis

1.2.1 Weak topology

In this subsection, we recall some basic definition and proportion of weak topology, which is a very important definition in analysis. One very useful technique in analysis is the argument of compactness : by compactness, we know the existence of a limit for a series, and then we can use other information to identify the uniqueness and characterize the limit in concrete example. However, we know that for a infinite Hilbert space H, the unit ball in not compact. Thus, we try other weaker topology instead. That is one useful application of weak topology.

In the following, let H be a Hilbert space and H' its dual (linear bounded functional of H) and $\langle \cdot, \cdot \rangle_{H,H'}$ its duality bracket. Sometimes we also abuse a little the notation to write $f(x)$ as $\langle x, f \rangle_{H,H'}$ of $x \in H$ and $f \in H'$.

Definition 1.2.1 (Weak convergence and weak topology)**.** We say a family of element $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a limit x and we denote it by $x_n \rightharpoonup x$ if and only if

$$\forall f \in H', \lim_{n\to\infty} \langle x_n, f \rangle_{H,H'} = \langle x, f \rangle_{H,H'} .$$

The weak topology is the the topology generated by the open set induced by all the functional $f \in H'$.

A first very elementary property is that the weak topology is weaker than the topology induced by the norm $\|\cdot\|_H$. Here we list some more elementary properties of the weak topology.

Proposition 1.2.1. *1. If $x_n \rightharpoonup x$ in H, then $\|x\|_H \leqslant \liminf_{n\to\infty} \|x_n\|_H$.*

2. If $x_n \rightharpoonup x$ in H and $f_n \to f$ in H', then we have

$$\lim_{n\to\infty} \langle x_n, f_n \rangle_{H,H'} = \langle x, f \rangle_{H,H'} .$$

The first one has the similar form as the Fubini lemma.

Compared to the topology of the norm, one advantage is the compactness of the weak topology.

Theorem 1.2.1 (Banach Alouglu)**.** *Every unit ball in H is compact in weak topology.*

In the Hilbert space context, thanks to the inner product structure $\langle \cdot, \cdot \rangle_H$, we have one more proposition

Proposition 1.2.2. *Let H be a Hilbert space and $x_n \rightharpoonup x$, then*

$$x_n \xrightarrow{\|\cdot\|_H} x \iff \lim_{n\to\infty} \|x_n\|_H = \|x\|_H.$$

Remark. This proposition is interesting, since normally we cannot deduce the strong convergence from the weak convergence. But with one more condition, here we can do it in Hilbert space. Similar spirit also exists in other topic, like the Shaffé lemma in probability theory. Moreover, the quantitative analysis of homogenization also shares this idea : some convergence only depends on the convergence of one observable.

Finally, we recall the famous Riesz lemma.

Lemma 1.2.1 (Riesz)**.** *Let H be a Hilbert space and f a bounded linear functional in H', then there exists an element $x_f \in H$ such that*

$$\forall x \in H, \qquad \langle x, f \rangle_{H,H'} = \langle x, x_f \rangle_H .$$

1.3 Function spaces

We introduce some function space where we will work on for the solution of equation. The theory of function space is a basis of the study of PDE and there are many good references. However, in the study of homogenization, sometimes we do the scaling of the domain, so some a priori estimates have to be done with more carefulness. Especially, we hope to know whether the constant in some useful inequalities (Hölder, Sobolev embedding, Poincaré etc) depends on the size of the area, so we revise these inequalities and their application in weighted norm defined below. The part about Sobolev space is after [ES98, Chapter 5] and the part about fractional Sobolev space is after [DNPV12].

In the following, we use $\{e_1, e_2, \cdots e_d\}$ as the canonical basis of $\mathbb{R}^d$ and $|U|$ for the Lebesgue measure of Borel set $U \subset \mathbb{R}^d$. We use $\partial_{x_j} u$ to define the weak derivative in the sense of Schwartz, that is for every $\phi \in C_c^\infty(U)$, we have

$$\int_U \partial_{x_j} u \phi = - \int_U u \partial_{x_j} \phi.$$

For $\beta \in \mathbb{N}^d$, we define also a multi-index weak derivative $\partial^\beta f$ such that

$$|\beta| := \sum_{i=1}^d \beta_i \quad \text{and} \quad \partial^\beta f = \partial_{x_1}^{\beta_1} \cdots \partial_{x_d}^{\beta_d} f.$$

1.3.1 Banach space $L^p(U)$ and $C^{k,\alpha}(U)$

We introduce at first the classical $L^p(U)$ space and Hölder space $C^{k,\alpha}(U)$.

Definition 1.3.1 ($L^p(U)$ space)**.** For every $p \in [1, +\infty)$, we define a norm

$$\|f\|_{L^p(U)} = \left(\int_U |f(x)|^p \, dx \right)^{\frac{1}{p}},$$

and we denote by $L^p(U)$ all the Lebesgue measurable function f such that $\|f\|_{L^p(U)} < \infty$. For the case $p = \infty$, we define its norm as

$$\|f\|_{L^\infty(U)} = \inf\left\{x | m(\{|f| > x\}) > 0\right\}.$$

Under both case, the space $L^p(U)$ is a Banach space.

Since we often scale the domain in the theory of homogenization, we define also the weighted L^p space.

Definition 1.3.2 ($\underline{L}^p(U)$ space)**.** For every $p \in [1, +\infty]$, the weighted norm $\underline{L}^p(U)$ is defined for a bounded domain $U \subset \mathbb{R}^d$ as

$$\|f\|_{\underline{L}^p(U)} = \left(\frac{1}{|U|}\int_U |f(x)|^p \, dx\right)^{\frac{1}{p}} = |U|^{-\frac{1}{p}} \|f\|_{L^p(U)}.$$

The Lebesgue measurable functions bounded under this norm also form a Banach space.

The weighted norm has many nice property, for example, for $1 \leqslant p \leqslant q \leqslant \infty$ one direct application of Hölder inequality tells us

$$\|f\|_{L^p(U)} \leqslant \|f\|_{L^q(U)} |U|^{\frac{1}{p}-\frac{1}{q}},$$

and we use the definition of weighted norm to get

$$\|f\|_{\underline{L}^p(U)} \leqslant \|f\|_{\underline{L}^q(U)}.$$

This helps us to avoid the term of constant coming from the size of area U.

Definition 1.3.3 (Hölder space $C^{k,\alpha}(U)$)**.** For every $0 \leqslant \alpha \leqslant 1$, we define the semi-norm

$$[f]_{C^{0,\alpha}(U)} = \sup_{x,y\in U, x\neq y}\left\{\frac{|f(x) - f(y)|}{|x-y|^\alpha}\right\},$$

and the Hölder norm as

$$\|f\|_{C^{0,\alpha}(U)} = \|f\|_{C(U)} + [f]_{C^{0,\alpha}(U)}.$$

In more general case, for every $k \in \mathbb{N}, 0 \leqslant \alpha \leqslant 1$, we define

$$\|f\|_{C^{k,\alpha}(U)} = \sum_{|\beta|<k} \left\|\partial^\beta f\right\|_{C(U)} + \sum_{\beta=k} \left[\partial^\beta f\right]_{C^{0,\alpha}(U)}.$$

The continuous function with bounded Hölder norm also forms a Banach space and we denote it by $C^{k,\alpha}(U)$.

1.3.2 Convolution

For $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$ where $\frac{1}{p} + \frac{1}{q} = 1$, we denote the convolution of f and g by

$$f \star g(x) = \int_{\mathbb{R}^d} f(y) g(x-y) \, dy.$$

The convolution of function inherits good property from the two. For example, the convolution between a $L^1_{loc}(\mathbb{R}^d)$ and a $C_c^\infty(\mathbb{R}^d)$ function is of class $C^\infty(\mathbb{R}^d)$. In this article, two mollifiers used are the heat kernel $\Phi_r(x)$, defined for $r > 0$ and $x \in \mathbb{R}^d$ by

$$\Phi_r(x) := \frac{1}{(4\pi r^2)^{d/2}} \exp\left(-\frac{x^2}{4r^2}\right).$$

and the bump function $\zeta \in C_c^\infty(\mathbb{R}^d)$

$$\zeta(x) := c_d \exp(-(1-|x|^2)^{-1})\mathbf{1}_{\{|x|<1\}},$$

where c_d is the constant of normalization such that $\int_{\mathbb{R}^d} \zeta(x)dx = 1$. Finally, we use the notation

$$\zeta_\varepsilon(x) = \frac{1}{\varepsilon^d}\zeta\left(\frac{x}{\varepsilon}\right),$$

as a mollifier in scale $\varepsilon > 0$.

1.3.3 Sobolev space $W^{k,p}(U), W_0^{k,p}(U)$

In this subsection, we give the definition of Sobolev space. The theory of Sobolev space is very classical since we often try to find the solution of PDE in this space for the reason both in maths: it is a Banach space and separable for good index, we can use weak compactness to get a sub-sequence limit etc, and also for the reason in physics: sometimes the finite norm has a direct natural interpretation as finite energy in physical context.

Definition 1.3.4 (Sobolev space $W^{k,p}(U), W_0^{k,p}(U)$)**.**

1. For each $k \in \mathbb{N}, 1 \leqslant p \leqslant \infty$, we denote by $W^{k,p}(U)$ the classical Sobolev space on U equipped with the norm

$$\|f\|_{W^{k,p}(U)} = \sum_{0 \leqslant |\beta| \leqslant k} \left\|\partial^\beta f\right\|_{L^p(U)},$$

and $W^{k,p}(U)$ is the function space containing all the Lebesgue measurable function bounded under this norm.

2. We use $W_0^{k,p}(U)$ to define the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.

3. For each $-k \in \mathbb{N}$, we define also $W^{-k,p}(U)$ to be the dual of $W_0^{k,p'}(U)$ where $\frac{1}{p} + \frac{1}{p'} = 1$

$$\|f\|_{W^{-k,p}(U)} := \sup\left\{\int_U fg, g \in W_0^{k,p'}(U), \|g\|_{W_0^{k,p'}(U)} \leqslant 1\right\}.$$

Here, we abuse the use of notation $\int_U f(x)g(x)\,dx$, since the function space $H^{-1}(U)$ also contains linear functional, which is not necessarily a function.

4. In the case where $p = 2$, we also use $H^k(U)$ and $H_0^k(U)$ to refer respectively $W^{k,2}(U)$ and $W_0^{k,p}(U)$.

From the definition, we see that $C_c^\infty(U)$ is dense in the space $W_0^{k,p}(U)$, but this is not the case of $W^{k,p}(U)$. The problem comes from the boundary effect, intuitively, near the boundary, the function will always go down to 0 for functions in $W_0^{k,p}(U)$, so it can be approximated by the regular function with compact support. However, the function in $W^{k,p}(U)$ can have very sharp fluctuation near the boundary, so we have to give up the condition of compact support. In fact, we can use $C^\infty(U)$ to approximate the function space $W^{k,p}(U)$ rather than $C_c^\infty(U)$. See [ES98, Chapter 5.3].

There are some other similar questions whose answers are different for $W^{k,p}(U)$ and $W_0^{k,p}(U)$:

1. Can we see $W^{k,p}(U)$ as a subspace of $W^{k,p}(\mathbb{R}^d)$? The answer is positive for $W_0^{k,p}(U)$ but it depends on the regularity of ∂U for $W^{k,p}(U)$. The problem is still near the boundary, because in $W^{k,p}(U)$, the function does not have to have a sense for derivative at the boundary. If $\partial U \in C^1$, then we can do some reflection and define a function called *the extension of function* and we denote it by $Ext(\cdot)$. It satisfies that for every $f \in W^{1,p}$, $Ext(f) = f$ a.e in U and

$$\|Ext(f)\|_{W^{1,p}(\mathbb{R}^d)} \leqslant C(U,d) \|f\|_{W^{1,p}(U)}.$$

   See [ES98, Chapter 5.4] for more details.

2. Can we talk about the function restricted at the boundary ∂U ? If it makes sense, it is called *the trace of function* and we denote it by $T(\cdot)$. If we want a bounded embedding $T : W^{1,p}(U) \to L^p(U)$ i.e

$$\|T(f)\|_{L^p(\partial U)} \leqslant C(p,U) \|f\|_{W^{1,p}(U)},$$

   we have to add some regularity to the boundary and in fact the condition $\partial U \in C^1$ also suffices. See [ES98, Chapter 5.5] for the trace theorem.

We observe that when $k = 0$, the Sobolev space $W^{0,p}(U) = L^p(U)$. Moreover, we can also define a weighted Sobolev norm in the following sense.

Definition 1.3.5 (Weighted Sobolev norm)**.** When $|U| < \infty$, we also define the weighted norm for $k \in \mathbb{N}, 1 \leqslant p \leqslant \infty$ that

$$\|f\|_{\underline{W}^{k,p}(U)} := \sum_{0 \leqslant |\beta| \leqslant k} |U|^{\frac{|\beta|-k}{d}} \|\partial^\beta f\|_{\underline{L}^p(U)}.$$

Finally, we remark that one advantage of the definition of $\underline{W}^{k,p}$ is that it is consistent with the scaling of the Poincaré inequality (see [ES98]) i.e if U has $C^{1,1}$ boundary

$$\forall f \in W_0^{1,p}(U), \|f\|_{\underline{H}^{1,p}(U)} \leqslant C(d) \|\nabla f\|_{\underline{L}^p(U)}.$$

1.3.4 Embedding theory

One important property is the relationship between different function spaces of different parameters. A first important theorem is the interpolation of $L^p(U)$ space.

Theorem 1.3.1 (L^p interpolation)**.** *For $1 \leqslant p \leqslant r \leqslant q \leqslant \infty$ such that*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},$$

we have $L^p(U) \cap L^q(U) \subset L^r(U)$ i.e

$$\forall f \in L^p(U) \cap L^q(U), \qquad \|f\|_{L^r(U)} \leqslant C(p,q,r) \|f\|^{\theta}_{L^p(U)} \|f\|^{1-\theta}_{L^r(U)}.$$

We can use both Marcinkiewicz interpolation theorem or Riesz–Thorin theorem to obtain the result, see [Gra08, Chapter 1] for the its proof.

For the embedding theory, the main tool is Nirenberg-Sobolev inequality and Morrey's inequality, which says the comparison between different norms.

Proposition 1.3.1 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Assume that $1 \leqslant p < d$, then we define its Sobolev conjugate p^**

$$p^* = \frac{dp}{d-p} \iff \frac{1}{p^*} - \frac{1}{d},$$

then there exists a constant $C(d,p)$ such that for every $f \in C_c^\infty(\mathbb{R}^d)$

$$\|f\|_{L^{p^*}} \leqslant C(d,p) \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

Since this inequality is done in the whole space, it does not involves the constant of the size of the support. Using the density of $C_c^\infty(\mathbb{R}^d)$ in $W_0^{1,p}(U)$ and the extension theorem, we obtain the following corollaries.

Corollary 1.3.1.

1. *For every $f \in W_0^{1,p}(U)$ and every $q \in [1,p^*]$, we have $W_0^{1,p}(U) \hookrightarrow L^{p^*}(U)$ and*

$$\|f\|_{L^q(U)} \leqslant |U|^{\frac{1}{q}-\frac{1}{p^*}} \|f\|_{L^{p^*}} \leqslant C(d,p)|U|^{\frac{1}{q}-\frac{1}{p^*}} \|\nabla f\|_{L^p(U)}.$$

2. *For every $U \subset \mathbb{R}^d$ with $\partial U \in C^1$, then for every $f \in W^{1,p}(U)$ and every $q \in [1,p^*]$, we have $W^{1,p}(U) \hookrightarrow L^{p^*}(U)$ and*

$$\|f\|_{L^q(U)} \leqslant |U|^{\frac{1}{q}-\frac{1}{p^*}} \|f\|_{L^{p^*}} \leqslant C(d,p,U)|U|^{\frac{1}{q}-\frac{1}{p^*}} \|f\|_{W^{1,p}(U)}.$$

In the case $p > d$, as the Kolmogorov inequality, it implies the Hölder continuity.

Proposition 1.3.2 (Morrey's inequality)**.** *For every $f \in C^1(\mathbb{R}^d)$, we have an inequality that*

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)-f(x)|\,dy \leqslant C(d,p)\int_{B_r(x)} \frac{|\nabla f(y)|}{|y-x|^{d-1}}\,dy.$$

So in the case $1 < p < \infty$, we have

$$\begin{aligned} \sup_{\mathbb{R}^d} |f| &\leqslant C(d,p)\|f\|_{W^{1,p}(\mathbb{R}^d)} \\ [f]_{C^{0,1-d/p}(\mathbb{R}^d)} &\leqslant C(d,p)\|\nabla f\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

These imply that $W^{1,p}(U) \hookrightarrow C^{0,1-d/p}(U)$.

We apply the arguments above by induction and can get the generalized embedding theorem.

Theorem 1.3.2 (Embedding theorem)**.**

1. *For every $1 \leqslant p < \frac{d}{k}$, we denote by*

$$p^* = \frac{dp}{d-pk} \iff \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}.$$

Then we have embedding $W^{k,p}(U) \hookrightarrow L^{p^*}(U)$ that

$$\|f\|_{L^{p^*}(U)} \leqslant C(d,k,p,U)\|f\|_{W^{k,p}(U)}.$$

2. *If* $k > \frac{d}{p}$, *then* $f \in C^{k-\left[\frac{d}{p}\right]-1,\gamma}(\bar{U})$ *where*

$$\gamma = \begin{cases} \left[\frac{d}{p}\right] + 1 - \frac{d}{p}, \frac{d}{p} \text{ is not an integer} \\ \text{any positive number} < 1, \frac{d}{p} \text{ is an integer.} \end{cases}$$

And we have an estimate

$$\|f\|_{C^{k-\left[\frac{d}{p}\right]-1,\gamma}(\bar{U})} \leqslant C(d,k,p,\gamma,U) \|f\|_{W^{k,p}(U)}$$

In the case $q < p^*$, the embedding theory can be better.

Theorem 1.3.3 (Rellich)**.** *For any* $U \subset \mathbb{R}^d$ *open bounded with* $\partial U \in C^1$ *and* $1 \leqslant q < p^*$, *we have* $W^{1,p}(U) \hookrightarrow L^q(U)$ *and this embedding is compact.*

1.3.5 Fractional Sobolev space $W^{\alpha,p}(U)$

Besides the classical integer order Sobolev space, we can also define fractional order Sobolev space $W^{\alpha,p}(U)$ and the Sobolev embedding can be also generalized in fractional case.

Definition 1.3.6 (Fractional Sobolev space)**.** For every $0 < \alpha < 1$, we define a seminorm

$$[f]^p_{W^{\alpha,p}} := (1-\alpha) \int_U \int_U \frac{|f(x)-f(y)|^p}{|x-y|^{d+\alpha p}} \, dxdy,$$

and we define the norm $W^{k+\alpha,p}(U)$

$$\|f\|_{W^{k+\alpha,p}(U)} := \sum_{|\beta| \leqslant k} \left\|\partial^\beta f\right\|_{L^p(U)} + \sum_{|\beta|=k} \left[\partial^\beta f\right]_{W^{\alpha,p}(U)}.$$

Then, with a little abuse of notation, $W^{k+\alpha,p}(U)$ also refers to the function space containing the Lebesgue measurable function bounded under this norm.

We have a similar result of embedding theorem.

Theorem 1.3.4 (Generalized embedding theorem)**.** *For every* $0 < \beta < \alpha < \infty$ *and* $1 \leqslant p, q \leqslant \infty$ *satisfying* $\beta - \frac{d}{q} \leqslant \alpha - \frac{d}{p}$, *then we have* $W^{\alpha,p}(U) \hookrightarrow W^{\beta,q}(U)$.

If we take $\beta = 0$, this theorem says $q \leqslant p^* = \frac{dp}{d-\alpha p}$, we have the embedding, which is as the classical embedding theorem except replacing the entire order derivative with that of fractional order. The proof of this theorem is technique and we have to treat the case "Fractional $\hookrightarrow$ Fractional", "Entire $\hookrightarrow$ Fractional" "Fractional $\hookrightarrow$ Entire". See [DNPV12] for complete proofs.

1.4 Basic theory of elliptic equation

In this part, we give a short resume about the general theory of second order elliptic equation, especially the equation of divergence form and harmonic equation - the two most typical equations. We will focus on the most basic properties of the equation : existence, uniqueness, the regularity with respect to the data, the coefficients and the domain. The fine results can be found in many wonderful references like [GT15], [HL11] and Section 1.4 mainly bases on the lecture note [Bab] and [All05].

1.4.1 Existence and uniqueness

In the following, we study the equation of divergence form defined in eq. (1.1.2)

$$\begin{cases} -\nabla \cdot \mathbf{a}\nabla u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{1.4.1}$$

By the principle of superposition, we can assume without loss of generality that $g = 0$. We try to find the solution u in the space $H^1(U)$ and also assume that $f \in H^{-1}(U)$ and the coefficient $\mathbf{a} \in L^\infty(U)$ and satisfies the *uniform ellipticity condition* that

$$\forall x, \xi \in \mathbb{R}^d, \exists \Lambda > 0, \text{ such that } \Lambda^{-1}|\xi|^2 \leqslant \xi \cdot \mathbf{a}(x)\xi \leqslant \Lambda|\xi|^2. \tag{1.4.2}$$

and we try to find its *weak solution* i.e

$$\forall \phi \in H^1_0(U), \qquad \int_U \nabla u \cdot \mathbf{a}\nabla\phi = \int_U f\phi. \tag{1.4.3}$$

Remark. The frame of weak solution is one important topic in modern PDE theory. The motivation is that some explicit formula like in classical heat equation or wave equation is not accessible in general, so the first goal is sometimes to make sure the existence so that we can talk about a solution of the equation. We hope, of course, the solution behaves like classical sense and we can apply the classical derivatives. However, on the one hand, sometimes some solution has discontinuity (like that of transport equation, where the discontinuity can propagate), on the other hand, if we add too many constrains to the solution, it is hard to talk about the existence of the solution. Therefore, one idea is to relax sometimes the condition to find a (weak) solution in a larger space and then study its regularity.

There are two main frames that can be used to attack the existence and uniqueness of the equation - by the theorem of Lax-Milgram and by the variational analysis.

Lax-Milgram frame

The Lax-Milgram theorem is used to treat the existence of the solution in Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Let a be a bilinear mapping and $L \in H'$ that

$$a : H \times H \to \mathbb{R}, \quad L : H \to \mathbb{R}.$$

Then we try to find a solution $u \in H$ such that

$$\forall v \in H, \quad a(u, v) = L(v). \tag{1.4.4}$$

To obtain the existence and uniqueness, we should add some conditions on the bilinear mapping. The Lax-Milgram theorem says the condition "continuity + coercivity" suffices.

Theorem 1.4.1 (Lax-Milgram)**.** *In the setting above, if $a(\cdot,\cdot)$ satisfies the following two condition:*

1. *Continue : there exists $M > 0$, such that for every $u, v \in H, |a(u, v)| \leqslant M \|u\|_H \|v\|_H$.*
2. *Coercive : there exists $m > 0$, such that for every $v \in H, a(v, v) \geqslant m \|v\|^2_H$.*

Then eq. (1.4.4) admits a unique solution $u \in H$ and we have

$$\|u\|_H \leqslant \frac{1}{m} \|L\|_{H'}. \tag{1.4.5}$$

See [Bab, Theorem 3.3.1] for its proof. We can check easily that in the weak solution frame of eq. (1.4.1) and the function space $H_0^1(U)$, the left hand side of eq. (1.4.3) is a continue, coercive bilinear mapping thanks to the uniform ellipticity condition eq. (1.4.2), and the right hand side is a linear form since $f \in H^{-1}(U)$. Thus, we obtain the existence and uniqueness.

Remark. In the case that $a(\cdot,\cdot)$ is symmetric, the proof of Lax-Milgram theorem is simpler since $a(\cdot,\cdot)$ defines a new inner product on H. In fact, we can use directly Riesz representation to obtain the same result.

Variational analysis frame

A second frame is variantional analysis. We state its main idea : the weak solution of eq. (1.4.3) is equivalent to minimize the functional

$$J(v) = \frac{1}{2}\int_U \nabla v \cdot \mathbf{a}\nabla v - \int_U f v. \tag{1.4.6}$$

Some analysis tells us that to obtain $\inf_{v\in H_0^1(U)} J(v) = \lim_{n\to\infty} J(v_n)$, we should have that $\{v_n\}_{n\in\mathbb{N}}$ in a bounded ball in $H^1(U)$. Then the weak compactness of Hilbert space applies and we have a weak limit u^* that

$$v_n \rightharpoonup u^*.$$

Moreover, from Proposition 1.2.1 and Theorem 1.3.3 we know that $J(u^*) \leqslant \liminf_{n\to\infty} J(v_n)$. So $u = u^*$ is the minimizer and it is the unique weak solution of eq. (1.4.3).

1.4.2 Harmonic function

Harmonic function satisfies

$$\begin{cases} -\Delta u = 0 & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{1.4.7}$$

If we suppose that $g \in C^2(\bar{U})$, we apply the theory in divergence form and get the existence of u. In fact, the property of u is nicer than the divergence form for two reasons : it does not have a source and it does not have perturbations of the coefficient $\mathbf{a}$. In one word, its regularity is very good. We illustrate it in the following aspect.

Harmonic function is C^∞

One of the most important property of harmonic function is the mean-value formula.

Proposition 1.4.1. *For every u harmonic in U and for every $\overline{B_r(x)} \subset U$, we have*

$$u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,dy = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,dy. \tag{1.4.8}$$

By the argument of induction, we get one corollary.

Corollary 1.4.1. *For every u satisfying eq. (1.4.8), then u is harmonic in U and $u \in C^\infty(U)$.*

Weak solution is strong solution

A second respect to show the good regularity of harmonic function is that the weak solution and strong solution coincide.

Proposition 1.4.2 (Weyl). *Given $u \in L^1_{loc}(U)$ such that*

$$\forall \phi \in C_c^\infty(U), \qquad \int_U u\Delta\phi = 0,$$

then after a modification of value at a zero measure set, $u \in C^\infty(U)$ and is harmonic on U.

However, the weak solution does not have such good regularity for the solution of an equation of divergence form. We will see it later.

Maximum principle

Proposition 1.4.3 (Maximum principle). *Given u harmonic in U, then in $\bar{U}$, the maximum of $|u|$ is always attained at the boundary that*

$$\max_{x\in\bar{U}} |u|(x) = \max_{x\in\partial U} |u|(x).$$

This proposition also proves the uniqueness of Dirichlet problem.

Liouville theorem

Finally, we discuss the harmonic function in whole space $\mathbb{R}^d$. One simple result of the mean value principle is that

Theorem 1.4.2 (Liouville). *If u is a bounded harmonic function in $\mathbb{R}^d$, then it is constant.*

In fact, this Liouville theorem can be done better : we can classify the harmonic function in whole space with its speed of increment of polynomial, and with polynomial speed increment, the dimension is finite. We state the result.

Theorem 1.4.3 (Liouville). *We define that*

$$\mathcal{P}_k = \left\{ u \in H^1_{loc}(\mathbb{R}^d), \Delta u = 0, \lim_{r\to\infty} r^{-(k+1)} \|u\|_{\underline{L}^2(B_r(0))} = 0 \right\}, \tag{1.4.9}$$

then $\mathcal{P}_k$ is a vector space of polynomial and

$$dim(\mathcal{P}_k) = \binom{d+k-1}{k} + \binom{d+k-2}{k-1}. \tag{1.4.10}$$

This theorem is remarkable since the space of harmonic function is of infinite dimension, but if we specify its speed, it becomes a finite subspace. A very simple example is that the harmonic function with linear increment has a basis $\{x_1, x_2, \cdots, x_d\}$ which increases towards d different directions.

A final remark is that the harmonic function is not only C^∞ but also analytic, moreover we can use the harmonic polynomial $\mathcal{P}_k$ to realize a harmonic approximation.

Proposition 1.4.4 (Harmonic approximation). *Given w a harmonic function in U, then $\forall k \in \mathbb{N}, \forall x \in U$ and $r > 0$ such that $\overline{B(x)} \subset U$, there exists a polynomial $p \in \mathcal{P}_k$ such that for every $t \in (0, \frac{r}{2}]$, we have*

$$\|w - p\|_{L^\infty(B_t(x))} \leqslant C_k(d) \left(\frac{t}{r}\right)^{k+1} \|w - p\|_{\underline{L}^2(B_r(x))}. \tag{1.4.11}$$

1.5 Notation $\mathcal{O}_s$

The definition of $\mathcal{O}$ is much used in the lecture note of [AKM18]. Its goal is to allow us to manipulate some small random errors as that of usual $\mathcal{O}$ notation. The definition of $\mathcal{O}_s$ is

$$X \leqslant \mathcal{O}_s(\theta) \iff \mathbb{E}\left[\exp((\theta^{-1}X)_+^s)\right] \leqslant 2, \tag{1.5.1}$$

where $(\theta^{-1}X)_+$ means $\max\{\theta^{-1}X, 0\}$. It could be used to calibrate a random error and has many good properties. One could use the Markov inequality to obtain that

$$X \leqslant \mathcal{O}_s(\theta) \Longrightarrow \forall x > 0, \mathbb{P}[X \geqslant \theta x] \leqslant 2\exp(-x^s),$$

so it gives an estimate of tail. Moreover, we could obtain the same estimate of the sum of a series of random variables although we do not know its joint distribution : for a measure space $(E, \mathcal{S}, m)$ and $\{X(z)\}_{z\in E}$ a family of random variables, we have

$$\forall z \in \mathcal{O}_s(E), X(z) \leqslant \mathcal{O}_s(\theta(z)) \Longrightarrow \int_E X(z)m(dz) \leqslant \mathcal{O}_s\left(C_s \int_E \theta(z)m(dz)\right), \tag{1.5.2}$$

where $0 < C_s < \infty$ is a constant and $C_s = 1$ for $s \geqslant 1$. See Appendix of [AKM18] for proofs and other operations on $\mathcal{O}_s$.

Chapter 2

Abstract qualitative theory

In this part, we recall some theory of abstract qualitative homogenization theory. The goal is to study the question asked in Section 1.1, especially in the equation

$$\begin{cases} -\nabla \cdot \mathbf{a}^{\varepsilon} \nabla u^{\varepsilon} = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{2.0.1}$$

, where $U \subset \mathbb{R}^d$ and is open with C^1 boundary and $f \in H^{-1}(U)$. We want to study its behavior when $\varepsilon \to 0$. This chapter is called a "abstract qualitative theory" since we sometimes prove the existence of a limit by some argument of weak compactness or Birkhoff ergodic theory and some quantity of the typical size of error is missing. That is why we develop next chapter about quantitative theory. The study of homogenization theory in maths goes back to [PBL78] and there are many good references and lectures on this topic. See [JKO12], [ALL10] etc. This chapter mainly base on [Pra16a], [Pra16b] and [PS08].

2.1 H-convergence

In the following, we always suppose that $\mathbf{a}$ satisfies the uniform ellipticity condition that $\forall x, \xi \in \mathbb{R}^d, \exists \Lambda > 0$ such that

$$\Lambda^{-1}|\xi|^2 \leqslant \xi \cdot \mathbf{a}(x)\xi \leqslant \Lambda|\xi|^2.$$

, but $\mathbf{a}$ can be of *periodic case* or *stochastic case*. In periodic case, we suppose that the period is 1, while in stochastic case, we suppose that

1. *Stationarity* $\forall A \in \mathcal{F}$
$$\mathbb{P}[T_y A] = \mathbb{P}[A].$$

2. *Unit range correlation*
$$\forall U, V \in \mathcal{B}(\mathbb{R}^d), d_H(W, V) > 1 \iff \mathcal{F}_W, \mathcal{F}_V \text{ are independent.}$$

   Here d_H is the Hausdorff distance in $\mathbb{R}^d$ and $\mathcal{F}_W, \mathcal{F}_V$is the σ-algebra generated by
$$\int_{\mathbb{R}^d} \chi \mathbf{a}_{i,j}, \text{ where } i, j \in \{1, 2, 3 \cdots d\}, \chi \in C_c^{\infty}(V).$$

We state the theorem.

Theorem 2.1.1 (H-convergence)**.** *In the setting of eq.* (2.0.1) *and both periodic or stochastic, we have a H-convergence : there exists a constant conductance matrix* $\bar{\mathbf{a}}$ *and a homogenized solution* $\bar{u}$ *satisfying*

$$\begin{cases} -\nabla \cdot \bar{\mathbf{a}} \nabla \bar{u} = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{2.1.1}$$

such that we can extract a sub-sequence of u^ε *and have*

1. $u^\varepsilon \xrightarrow[\varepsilon \to 0]{L^2(U)} \bar{u}$.

2. $\nabla u^\varepsilon \xrightharpoonup[\varepsilon \to 0]{L^2(U)} \nabla \bar{u}$.

3. $\mathbf{a}^\varepsilon \nabla u^\varepsilon \xrightharpoonup[\varepsilon \to 0]{L^2(U)} \bar{\mathbf{a}} \nabla \bar{u}$.

4. $\int_U \nabla u^\varepsilon \cdot \mathbf{a}^\varepsilon \nabla u^\varepsilon \xrightarrow{\varepsilon \to 0} \int_U \nabla \bar{u} \cdot \bar{\mathbf{a}} \nabla \bar{u}$.

(In the stochastic case, the theorem is stated in the sense almost surely.)

Existence of limit. There is some easy part of the theorem. By Lax-Milgram theorem, we have for every $\varepsilon > 0$,

$$\|u^\varepsilon\|_{H^1(U)} \leqslant \Lambda \|f\|_{H^{-1}(U)}.$$

Thus, by weak compactness in Hilbert space, up to an extraction, there exists $\bar{u} \in H^1_0(U)$ such that

$$u^\varepsilon \xrightharpoonup[\varepsilon \to 0]{H^1(U)} \bar{u}.$$

Moreover, by the Rellich theorem, this implies strong convergence in $L^2(U)$

$$u^\varepsilon \xrightarrow[\varepsilon \to 0]{L^2(U)} \bar{u}.$$

Same argument also assures that $\mathbf{a}^\varepsilon \nabla u^\varepsilon \xrightharpoonup[\varepsilon \to 0]{L^2(U)} \xi$. The main difficulty may be how to identify the limit $\bar{u}$. Even though we had proved $\xi = \bar{\mathbf{a}} \nabla \bar{u}$ for some $\bar{\mathbf{a}}$, the forth convergence $\int_U \nabla u^\varepsilon \cdot \mathbf{a}^\varepsilon \nabla u^\varepsilon \xrightarrow{\varepsilon \to 0} \int_U \nabla \bar{u} \cdot \bar{\mathbf{a}} \nabla \bar{u}$ is not free. In fact, we have weak convergence of ∇u^ε and $\mathbf{a}^\varepsilon \nabla u^\varepsilon$, but how we get the convergence of its product ? □

The answer is reduced to the following lemma.

2.1.1 Div-Curl theorem

The following theorem is the key of the proof of the qualitative homogenization theory. It appears at first in at [Mur78] and also found in [Eva90]. See [Pra16a] for a detailed discussion and we also adopt the version from it. Here, for a vector $v \in \mathbb{R}^d$, the notation of divergence and curl $\nabla \cdot v$ and $\nabla \times v$ are defined as the common sense.

$$\begin{aligned} \nabla \cdot v &= \sum_{i=1}^{d} \partial_{x_i} v_i, \\ (\nabla \times v)_{i,j} &= \partial_{x_j} v_i - \partial_{x_i} v_j. \end{aligned}$$

Theorem 2.1.2 (Div-Curl theorem.). *For every $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, given two sequences of vectors $\{v_n\}_{n\geqslant 0}, \{w_n\}_{n\geqslant 0}$ satisfying*

$$\|v_n\|_{L^p(U)} + \|\nabla \cdot v_n\|_{L^p(U)} \text{ uniformly bounded, } v_n \xrightarrow[n\to\infty]{L^p(U)} v,$$
$$\|w_n\|_{L^{p'}(U)} + \|\nabla \times w_n\|_{L^{p'}(U)} \text{ uniformly bounded, } w_n \xrightarrow[n\to\infty]{L^{p'}(U)} w,$$

then we have

$$v_n \cdot w_n \xrightarrow[n\to\infty]{D'(U)} v \cdot w,$$

where $D'(U)$ refers the space of distribution, namely the dual of $D(U) = C_c^\infty(U)$.

In our example, we will apply this theorem with one term of divergence free and one term with curl free.

2.1.2 Periodic case

We solve at first an equation in $L^2(\mathbb{T}^d)$ that

$$-\nabla \cdot \mathbf{a}^T(e + \nabla\phi_e) = 0, \tag{2.1.2}$$

up to a constant and we call ϕ_e *the first order corrector* associated to e. It plays an important role in the theory of homogenization. Since ϕ_e is periodic, we have naturally

$$\int_{\mathbb{T}^d} \nabla\phi_e = 0.$$

Then, thanks to the periodic, we have that

$$\mathbf{a}^T\left(\frac{\cdot}{\varepsilon}\right)\left(e + \nabla\phi_e\left(\frac{\cdot}{\varepsilon}\right)\right) \xrightarrow[\varepsilon\to 0]{L^2(U)} \int_{\mathbb{T}^d} \mathbf{a}^T(e + \nabla\phi_e), \qquad \nabla\phi_e\left(\frac{\cdot}{\varepsilon}\right) \xrightarrow[\varepsilon\to 0]{L^2(U)} 0. \tag{2.1.3}$$

A nice identity is that

$$\nabla u^\varepsilon \cdot \mathbf{a}^T\left(\frac{\cdot}{\varepsilon}\right)\left(e + \nabla\phi_e\left(\frac{\cdot}{\varepsilon}\right)\right) = \mathbf{a}^\varepsilon \nabla u^\varepsilon \cdot \left(e + \nabla\phi_e\left(\frac{\cdot}{\varepsilon}\right)\right).$$

We observe that, the product on both left hand side and right hand side is of the form "one term of divergence free times one term of curl free". Therefore, when we pass the weak limit of an extraction, we get the identity

$$\nabla\bar{u} \cdot \int_{\mathbb{T}^d} \mathbf{a}^T(e + \nabla\phi_e) = \xi \cdot e.$$

We use the canonical basis to form an identity matrix that $\mathrm{Id} = (e_1, e_2, \cdots e_d)$ and we let $\Gamma = (\nabla\phi_{e_1}, \nabla\phi_{e_2}, \nabla\phi_{e_3} \cdots \nabla\phi_{e_d})^T$. Then we deduce from the above identity

$$\left(\int_{\mathbb{T}^d} (\mathrm{Id} + \Gamma)\mathbf{a}\right) \nabla\bar{u} = \xi.$$

We define that

$$\bar{\mathbf{a}} := \left(\int_{\mathbb{T}^d} (\mathrm{Id} + \Gamma)\mathbf{a}\right), \tag{2.1.4}$$

in the periodic case. Then we have that $\xi = \bar{\mathbf{a}}\nabla\bar{u}$. Moreover, We test the identity above with $\varphi \in C_c^\infty(U)$, and we get that

$$\begin{aligned}\int_U \nabla\varphi \cdot \bar{\mathbf{a}}\nabla\bar{u} &= \int_U \nabla\varphi \cdot \xi \\ &= \lim_{\varepsilon\to 0}\int_U \nabla\varphi \cdot \mathbf{a}^\varepsilon\nabla u^\varepsilon \\ &= \lim_{\varepsilon\to 0}\int_U \varphi f.\end{aligned}$$

We deduce from it that $-\nabla \cdot \bar{\mathbf{a}}\nabla\bar{u} = f$.

Finally, we observe easily that the convergence of energy

$$\int_U \nabla u^\varepsilon \cdot \mathbf{a}^\varepsilon \nabla u^\varepsilon \xrightarrow{\varepsilon\to 0} \int_U \nabla\bar{u}\cdot\bar{\mathbf{a}}\nabla\bar{u},$$

is also the result of the div-curl theorem and approximation of $\mathbf{1}_U$ by functions in $C_c^\infty(U)$.

2.1.3 Stochastic case

In fact, all the proof in the stochastic case follows from that in periodic case. Except the existence of first order corrector needs careful treatment. See [JKO12, Theorem 7.2]. We admit its existence and force that $\mathbb{E}[\nabla\phi_e] = 0$, then in the step eq. (2.1.3), we have a similar result after Birkhoff ergodic convergence that

$$\mathbf{a}^T\left(\frac{\cdot}{\varepsilon}\right)\left(e + \nabla\phi_e\left(\frac{\cdot}{\varepsilon}\right)\right) \xrightarrow[\varepsilon\to 0]{L^2(U)} \mathbb{E}\left[\mathbf{a}^T(e+\nabla\phi_e)\right], \qquad \nabla\phi_e\left(\frac{\cdot}{\varepsilon}\right) \xrightarrow[\varepsilon\to 0]{L^2(U)} 0. \tag{2.1.5}$$

This modification implies the definition

$$\bar{\mathbf{a}} := \mathbb{E}\left[(\mathrm{Id}+\Gamma)\mathbf{a}\right], \tag{2.1.6}$$

and all the other steps of proof are exactly the same.

2.2 Two-scale expansion

Another classical idea to study the problem of homogenization is to use the perturbation of operator, that one can assume that the operator $\mathbf{a}^\varepsilon$ has an expansion

$$\mathbf{a}^\varepsilon = \frac{1}{\varepsilon^2}\mathbf{a}_0 + \frac{1}{\varepsilon}\mathbf{a}_1 + \mathbf{a}_2\cdots$$

and the solution u^ε has also an expansion

$$u^\varepsilon = u_0\left(x,\frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x,\frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x,\frac{x}{\varepsilon}\right)\cdots$$

and then try to establish some relations between these equations. The main spirit of this idea is that the function u^ε has a good asymptotic form in global sense but is very zigzag locally. So, we imagine that there exists a function $u : U\times\mathbb{T}\to\mathbb{R}$, where the first variable controls the global information while the second controls the local information. Then, the solution u^ε can be seen as a projection of this function u that

$$u^\varepsilon(x) = u\left(x,\frac{x}{\varepsilon}\right).$$

This idea is later translated in a rigorous mathematical theory of homogenization in periodic coefficients called two-scale expansion (or test with oscillating function) by [All92]. We follow [PS08, Chapter 19] for a nice introduction.

2.2.1 Two-scale convergence

We introduce at first the function space with two scales.

Definition 2.2.1 ($L^2(U, \mathbb{T}^d), L^2(U, H^1(\mathbb{T}^d)), L^2(U, C(\mathbb{T}^d))$)**.** For every $U \subset \mathbb{R}^d$ open set with Liptchitz boundary, we define a function space $L^2(U, \mathbb{T}^d)$ with the norm

$$\|f\|^2_{L^2(U,\mathbb{T}^d)} := \int_U \int_{\mathbb{T}^d} |f(x, y)|^2 \, dxdy.$$

A similar definition is that of $L^2(U, H^1(\mathbb{T}^d))$

$$\|f\|^2_{L^2(U,H^1(\mathbb{T}^d))} := \int_U \int_{\mathbb{T}^d} |\nabla_y f(x, y)|^2 \, dxdy.$$

The function norm $L^2(U, C(\mathbb{T}^d))$ is defined for the norm

$$\|f\|^2_{L^2(U,C(\mathbb{T}^d))} := \int_U \left(\sup_{y \in \mathbb{T}^d} |u(x, y)| \right)^2 dx.$$

The two-scale convergence can be seen as a weak convergence of $L^2(U, C(\mathbb{T}^d))$.

Definition 2.2.2 (Two-scale convergence)**.** $\{v^\varepsilon\}_{\varepsilon>0}$ is a series of functions of space $L^2(U)$, then we denote by $v^\varepsilon \xrightarrow[\varepsilon\to 0]{2} v_0$ the two-scale convergence if there is a function $v_0 \in L^2(U, \mathbb{T}^d)$ and for every $\varphi \in L^2(U, C(\mathbb{T}^d))$, we have

$$\lim_{\varepsilon\to 0} \int_U v^\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) \, dx = \int_U \int_{\mathbb{T}^d} v_0(x, y)\varphi(x, y) \, dxdy. \tag{2.2.1}$$

Remark. One observation is that if $v^\varepsilon(x) = v\left(x, \frac{x}{\varepsilon}\right)$ for some $v \in L^2(U, C(\mathbb{T}^d))$, we have directly

$$v^\varepsilon \xrightarrow[\varepsilon\to 0]{2} \int_{\mathbb{T}^d} v(\cdot, y) \, dy.$$

Therefore, this definition captures the idea of two different scales.

As the weak compactness in Hilbert space, the following weak convergence theorem for two-scale convergence is very useful.

Lemma 2.2.1 (Weak compactness of two-scale convergence)**.** *Given v^ε bounded in $H^1(U)$, then there exists $v_0 \in H^1(U), v_1 \in L^2(U, H^1(\mathbb{T}^d))$ such that the following is established :*

1. $v^\varepsilon \xrightarrow[\varepsilon\to 0]{H^1(U)} v_0$,
2. $v^\varepsilon \xrightarrow[\varepsilon\to 0]{2} v_0$,
3. $\nabla v^\varepsilon \xrightarrow[\varepsilon\to 0]{2} \nabla v + \nabla_y v_1$.

2.2.2 Main result

The main result of two-scale expansion is the following :

Theorem 2.2.1 (Two-scale expansion)**.** *In the setting of eq.* (2.0.1) *with periodic coefficients, there exists a solution $\bar{u}$ of homogenized equation such that we have a two-scale expansion*

$$w^{\varepsilon}(\cdot) := \bar{u}(\cdot) + \varepsilon \sum \partial_{x_i}\bar{u}(\cdot)\phi_{e_i}\left(\frac{\cdot}{\varepsilon}\right),$$

and it satisfies

$$\|u^{\varepsilon} - w^{\varepsilon}\|_{H^1(U)} \xrightarrow{\varepsilon\to 0} 0.$$

Proof. By the weak compactness lemma Lemma 2.2.1 we can easily find an candidate $(\bar{u}, u_1) \in H_0^1(U)\times L^2(U, H^1(\mathbb{T}^d))$ of the limit such that

$$u^{\varepsilon} \xrightarrow[\varepsilon\to 0]{H^1(U)} \bar{u}, \quad u^{\varepsilon} \xrightarrow[\varepsilon\to 0]{2} \bar{u},$$
$$\nabla u^{\varepsilon} \xrightarrow[\varepsilon\to 0]{2} \nabla\bar{u} + \nabla_y u_1.$$

The difficulty is to identify the limit $(\bar{u}, u_1)$. The main tool is still the Lax-Milgram theorem : we define the Hilbert space $X = H_0^1(U) \times L^2(U, H^1(\mathbb{T}^d))$ and we have $\mathbf{U} = (\bar{u}, u_1) \in X$ with the norm

$$\mathbf{U}_X^2 = \|\nabla\bar{u}\|_{L^2(U)}^2 + \|\nabla_y u_1\|_{L^2(U,\mathbb{T}^d)}^2.$$

Then, we define a bilinear mapping $a : X \times X \to$, for every $\Phi = (\varphi_0, \varphi_1) \in X$, we define

$$a(\Phi, \mathbf{U}) = \int_U \int_{\mathbb{T}^d} (\nabla\varphi + \nabla_y\varphi_1)\cdot \mathbf{a}(\nabla\bar{u} + \nabla_y u_1),$$

then $\mathbf{U}$ solves $a(\Phi, \mathbf{U}) = \langle\varphi_0, f\rangle_{L^2(U)}$ by verifying steps of passing to limit. The bilinear mapping is under the frame of Lax-Milgram, so we know its existence and uniqueness of solution. Moreover, by verifying

$$u_1(x,y) = (\phi_{e_1}, \phi_{e_2}, \phi_{e_3}\cdots\phi_{e_d})(y)\nabla\bar{u}(x)$$

we identify the limit is exact the solution of homogenized equation.

Finally, we calculate that

$$\begin{aligned}
\Lambda^{-1}\|\nabla u^{\varepsilon} - \nabla w^{\varepsilon}\|_{L^2(U)}^2 &\leqslant \left\langle \mathbf{a}^{\varepsilon}\left(\nabla u^{\varepsilon} - \bar{u}(\cdot) - \nabla_y u_1\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right), \nabla u^{\varepsilon} - \bar{u}(\cdot) - \nabla_y u_1\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right\rangle_{L^2(U)} \\
&\leqslant \langle \mathbf{a}^{\varepsilon}\nabla u^{\varepsilon}, \nabla u^{\varepsilon}\rangle_{L^2(U)} - \left\langle (\mathbf{a}^{\varepsilon} + \mathbf{a}^{\varepsilon T})\nabla u^{\varepsilon}, \bar{u}(\cdot) + \nabla_y u_1\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right\rangle_{L^2(U)} \\
&\quad + \left\langle \mathbf{a}^{\varepsilon}\left(\bar{u}(\cdot) + \nabla_y u_1\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right), \bar{u}(\cdot) + \nabla_y u_1\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right\rangle_{L^2(U)} \\
&= \langle f, u^{\varepsilon}\rangle_{L^2(U)} - \left\langle (\mathbf{a}^{\varepsilon} + \mathbf{a}^{\varepsilon T})\nabla u^{\varepsilon}, \bar{u}(\cdot) + \nabla_y u_1\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right\rangle_{L^2(U)} \\
&\quad + \left\langle \mathbf{a}^{\varepsilon}\left(\bar{u}(\cdot) + \nabla_y u_1\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right), \bar{u}(\cdot) + \nabla_y u_1\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right\rangle_{L^2(U)}
\end{aligned}$$

The three terms converge respectively to $a(\mathbf{U},\mathbf{U}), -2a(\mathbf{U},\mathbf{U}), a(\mathbf{U},\mathbf{U})$ by weak convergence or two-scale convergence, so we get the H^1 convergence. □

The two-scale expansion has many applications, for example, it provides an idea to approximate the heterogeneous equation from the homogenized equation, which can reduce the complexity of calculus. However, a direct adaption in stochastic coefficients case requires more work.

Chapter 3

Quantitative theory by variational analysis

In this chapter, we develop the theory of convergence in a quantitative version, since in Chapter 2 especially the case with stationary stochastic coefficients, we use the ergodic convergence theorem, which tells us nothing about the typical size of error. Thus, we lack the theoretical support when implementing numerical algorithm. It is until recently the quantitative analysis of homogenization obtains much progress. This chapter could be seen as a resume of [AKM18] but we do not want to take all the theorems and all the proofs in the lecture since it is self-contained book. We aim to make this chapter as a guide of book, aiming to get the main idea and understand some important and useful theorems in it.

In stead of focusing on the cell problem, in [AKM18] we take sometimes the viewpoint of the problem in large scale. For example, we define the area $\square_m = \left(-\frac{3^m}{2}, \frac{3^m}{2}\right)^d$ and try to solve the equation with boundary condition $l_p(x) = p \cdot x$

$$\begin{cases} -\nabla \cdot \mathbf{a}\nabla u = 0 & \text{in } \square_m, \\ u = l_p & \text{on } \partial\square_m. \end{cases} \tag{3.0.1}$$

We write the solution of this problem $u(\cdot, \square_m, p)$, then it relates closely to the solution of

$$\begin{cases} -\nabla \cdot \mathbf{a}^{3^{-m}} \nabla u^{3^{-m}} = 0 & \text{in } \square, \\ u^{3^{-m}} = l_p & \text{on } \partial\square, \end{cases} \tag{3.0.2}$$

by a scaling transform

$$u^{3^{-m}}(\cdot) = 3^{-m} u(3^m \cdot, \square_m, p). \tag{3.0.3}$$

The difference is that the equation eq. (3.0.1) provides a new viewpoint. In cell problem, we take $\varepsilon \to 0$ and here we take $m \to \infty$ with fixed affine boundary condition. Although the divergence form operator $-\nabla \cdot \mathbf{a}\nabla$ is not as good as that of harmonic $-\Delta$, we still hope that in large scale, it has some similar effect of harmonic equation and the solution $u(\cdot, \square_m, p)$ looks like the affine function l_p in some sense. The quantitative theory starts from this point and we will discuss it in Section 3.1. In fact, we can do better to get a complete Liouville theory like Theorem 1.4.3, which will be discussed in Section 3.3. Finally, we can also prove some quantitative version analysis of corrector and use them to get quantitative analysis of two-scale expansion (Section 3.4).

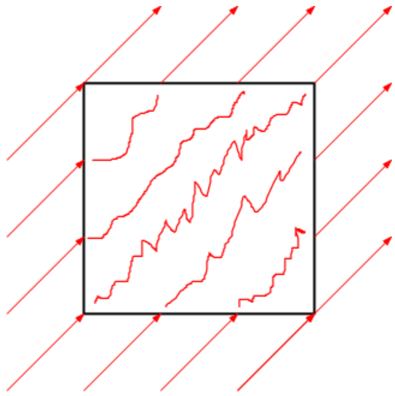

Figure 3.1: An image shows the behavior of the solution $u(\cdot, \square_m, p)$ of eq. (3.0.1). When we fix the affine boundary condition and enlarge the domain, the solution $u(\cdot, \square_m, p)$ looks like an affine function in large scale.

Section 3.1, Section 3.2, Section 3.3 and Section 3.4 resume the results respectively in [AKM18, Chapter 1, Chapter 2, Chapter 3 and Chapter 6]. We remark that for easily explaining the proof, the authors suppose that $\mathbf{a}$ is symmetric in these chapters, but we can also drop this condition by similar argument and more careful treatment in several steps. See [AKM18, Chapter 10] for this remark.

3.1 Quantitative version of H-convergence

In this part, we study a quantitative version of Theorem 2.1.1. As we have proved in Theorem 2.1.1, we know that the limit of eq. (3.0.2) is exactly l_p and we apply eq. (3.0.3) to get

$$\frac{1}{|\square_m|} \int_{\square_m} \frac{1}{2} \nabla u(\cdot, \square_m, p) \cdot \mathbf{a}(\cdot) \nabla u(\cdot, \square_m, p) \xrightarrow{m \to \infty} \frac{1}{2} p \cdot \bar{\mathbf{a}} p, \quad a.s.$$

If we interpret left hand side as an average energy, what we will prove in this section is that we can use the convergence of this quantity to measure the speed of convergence in the topology L^2 and weak H^1 etc.

3.1.1 Observable of average energy

We give at first some more precise description about the argument at the beginning and a heuristic analogue why it should work.

As we know, that solution of elliptic equation can be transformed to a problem of minimization. If we define *the unit area energy* that

$$\nu(U, p) := \inf_{v \in l_p + H_0^1(U)} \frac{1}{|U|} \int_U \frac{1}{2} \nabla v \cdot \mathbf{a} \nabla, \tag{3.1.1}$$

then its minimum is attained by the solution of elliptic equation eq. (3.0.1) for the same boundary condition but in domain $U \subset \mathbb{R}^d$ with Liptchitz boundary, and we denote it by $u(\cdot, U, p)$. This quantity has many interesting properties, the following two are easy to verify but very important.

Proposition 3.1.1. *1. Quadratic response. For every $w \in l_p + H_0^1(U)$, we have*

$$\frac{1}{2\Lambda|U|}\int_U |\nabla w - \nabla v|^2 \leqslant \frac{1}{|U|}\int_U \frac{1}{2}\nabla w \cdot \mathbf{a}\nabla w - \nu(U,p) \leqslant \frac{\Lambda}{2|U|}\int_U |\nabla w - \nabla v|^2. \quad (3.1.2)$$

2. Sub-additive The quantity ν is sub-additive i.e for a disjoint union $U = \sqcup_{i=1}^n U_i$, we have

$$\nu(U,p) \leqslant \sum_{i=1}^n \frac{|U_i|}{|U|}\nu(U_i,p). \quad (3.1.3)$$

Proof. The proof of eq. (3.1.2) is a direct calculus, while the proof of eq. (3.1.3) depends on a construction of sub-minimiser

$$\tilde{u}(\cdot,U,p) = \sum_{i=1}^n u(\cdot,U_i,p)\mathbf{1}_{U_i}, \quad (3.1.4)$$

since the boundary condition coincide. We test it in the eq. (3.1.1) and get the desired result. □

If we apply the result eq. (3.1.3) in the partition of $\square_m$

$$\nu(\square_m,p) \leqslant \sum_{z\in 3^{m-1}\mathbb{Z}^d\cap\square_m} \frac{|\square_{m-1}|}{|\square_m|}\nu(z+\square_{m-1},p).$$

We take expectation and use the stationarity to obtain that

$$\mathbb{E}[\nu(\square_m,p)] \leqslant \mathbb{E}[\nu(\square_{m-1},p)].$$

We adapt the notation here and the theorem Theorem 2.1.1 that

$$\nu(\square_m,p) \xrightarrow{m\to\infty} \frac{1}{2}p\cdot\bar{\mathbf{a}}p.$$

and by the sub-additive ergodic theory we also know that

$$\frac{1}{2}p\cdot\bar{\mathbf{a}}p = \lim_{m\to\infty}\mathbb{E}[\nu(\square_m,p)].$$

One interpretation is that $\nu(\square_m,p)$ serves as the unit area energy decreases to a limit. Our main theory of quantitative theory is that we can measure the H-convergence by the convergence of the unit area energy $\nu(\square_m,p)$. Readers can draw an analogue with the well known Schaffé's lemma in probability : A series of integrable random variables $\{X_n\}_{n\geqslant 0}$ converge almost surely to a integrable random variable X, then if $\mathbb{E}[X_n] \xrightarrow{n\to\infty} \mathbb{E}[X]$, we have $X_n \xrightarrow[n\to\infty]{L^1} X$. Therefore, the convergence of an observable does imply a stronger L^1 convergence.

Proposition 3.1.2 (Quantitative L^2)**.** *We define a quantity $\omega(n)$*

$$\omega(n) := \sup_{p\in B_1}\left(\mathbb{E}\left[\nu(\square_n,p)\right] - \frac{1}{2}p\cdot\bar{\mathbf{a}}p\right). \quad (3.1.5)$$

then there exists $0 < C(d,\Lambda) < \infty$ satisfying that for every $m \in \mathbb{N}$,

$$\mathbb{E}\left[\left|\nu(\square_m,p) - \frac{1}{2}p\cdot\bar{\mathbf{a}}p\right|\right] \leqslant C(d,\Lambda)\left(3^{-\frac{d}{4}m} + \omega\left(\left\lceil\frac{m}{2}\right\rceil\right)\right), \quad (3.1.6)$$

$$\mathbb{E}\left[3^{-2m}\left\|u(\cdot,\square_m,p) - l_p\right\|^2_{\underline{L}^2(\square_m)}\right] \leqslant C(d,\Lambda)\left(3^{-\frac{d}{4}} + \omega\left(\left\lceil\frac{m}{2}\right\rceil\right)\right). \quad (3.1.7)$$

The weak convergence can also be described by some quantity. However, we should give a good metric to describe the weak H^1 convergence. We define that

$$\|f\|_{\underline{\widehat{H}}^{-1}(U)} = \sup\left\{\left|\frac{1}{|U|}\int_U fg\right| : g \in H^1(U), \|g\|_{\underline{H}^1(U)} \leqslant 1\right\}$$

and we can prove that the topology under this norm is equivalent with the weak L^2 topology (See [AKM18, Exercise 1.4]) i.e

$$f_n \xrightharpoonup[n\to\infty]{L^2(U)} f \iff \lim_{n\to\infty} \|f_n - f\|_{\underline{\widehat{H}}^{-1}(U)} = 0.$$

So, the quantitative version of weak H^1 convergence is the following :

Proposition 3.1.3 (Quantitative weak H^1)**.** *We define another quantity*

$$\mathcal{E}(m) := \left(\sum_{n=0}^{m} 3^{n-m}\left(\frac{1}{|3^n\mathbb{Z}^d \cap \square_m|}\sum_{z\in 3^n\mathbb{Z}^d\cap\square_m}\sup_{p\in B_1}\left|\nu(z+\square_n, p) - \frac{1}{2}p\cdot\bar{\mathbf{a}}p\right|\right)^{\frac{1}{2}}\right), \tag{3.1.8}$$

then there exists $0 < C(d,\Lambda) < \infty$ *satisfying that for every* $m \in \mathbb{N}$

$$3^{-2m}\|\nabla u(\cdot,\square_m,p) - p\|^2_{\underline{\widehat{H}}^{-1}(\square_m)} + 3^{-2m}\|\mathbf{a}\nabla u(\cdot,\square_m,p) - \bar{\mathbf{a}}p\|^2_{\underline{\widehat{H}}^{-1}(\square_m)} \leqslant C(d,\Lambda)\left(3^{-2m} + \mathcal{E}(m)\right).$$

We see that $\mathcal{E}(m)$ is a weighted combination of some $\omega(n)$, and their quantities are random variables, which converge to 0 with a unknown speed. They will be discussed in a more details way in Section 3.2.

3.1.2 Renormalization group argument

The renormalization group argument says just we can divide the problem in some small scale and solve it. We start from the proof of Proposition 3.1.2.

Proof of first part Proposition 3.1.2. We apply at first the bias-variance decomposition

$$\begin{aligned}\mathbb{E}\left[\left|\nu(\square_m,p) - \frac{1}{2}p\cdot\bar{\mathbf{a}}p\right|\right] &\leqslant \mathbb{E}\left[|\nu(\square_m,p) - \mathbb{E}[\nu(\square_m,p)]|\right] + \mathbb{E}\left[\mathbb{E}[\nu(\square_m,p)] - \frac{1}{2}p\cdot\bar{\mathbf{a}}p\right] \\ &\leqslant 2\mathbb{E}\left[(\nu(\square_m,p) - \mathbb{E}[\nu(\square_m,p)])_+\right] + \omega(m)\end{aligned}$$

We use a quantity $\mathbb{E}[\nu(\square_n,p)], n\in\mathbb{N}, 0<n<m$ as an intermediate random variable and further develop the above inequality that

$$\begin{aligned}\mathbb{E}\left[\left|\nu(\square_m,p) - \frac{1}{2}p\cdot\bar{\mathbf{a}}p\right|\right] &\leqslant 2\mathbb{E}\left[(\nu(\square_m,p) - \mathbb{E}[\nu(\square_n,p)])_+\right] + 2(\mathbb{E}[\nu(\square_n,p)] - \mathbb{E}[\nu(\square_m,p)]) + \omega(m) \\ &\leqslant 2\mathbb{E}\left[(\nu(\square_m,p) - \mathbb{E}[\nu(\square_n,p)])_+\right] + 2\omega(n) + \omega(m) \\ \text{(Cauchy-Schwartz)} &\leqslant 2\mathbb{E}\left[(\nu(\square_m,p) - \mathbb{E}[\nu(\square_n,p)])_+^2\right]^{\frac{1}{2}} + 2\omega(n) + \omega(m)\end{aligned}$$

The following lemma is the key of the renormalization group argument and shows the spirit to manipulate different scales.

Lemma 3.1.1 (Renormalization inequality)**.** *There exists* $0 < C(d, \Lambda) < \infty$ *such that for all integer* $0 \leqslant n < m < \infty$

$$\mathbb{E}\left[(\nu(\square_m, p) - \mathbb{E}[\nu(\square_n, p)])_+^2\right] \leqslant C(d, \Lambda) 3^{-d(m-n)}. \tag{3.1.9}$$

Proof. By the sub-additive eq. (3.1.3), we have

$$\nu(\square_m, p) \leqslant \sum_{z \in 3^n \mathbb{Z}^d \cap \square_m} \frac{|\square_n|}{|\square_m|} \nu(z + \square_n, p) = \sum_{z \in 3^n \mathbb{Z}^d \cap \square_m} 3^{-d(m-n)} \nu(z + \square_n, p).$$

Since $(\cdot)_+^2$ is monotone, we put the right hand side quantity into the inequality that we want to control have the estimate

$$\begin{aligned}
\mathbb{E}\left[(\nu(\square_m, p) - \mathbb{E}[\nu(\square_n, p)])_+^2\right] &\leqslant \mathbb{E}\left[\left(3^{-d(m-n)} \sum_{z \in 3^n \mathbb{Z}^d \cap \square_m} \nu(z + \square_n, p) - \mathbb{E}[\nu(\square_n, p)]\right)_+^2\right] \\
&\leqslant 3^{-2d(m-n)} \times 3^d \sum_{z \in 3^n \mathbb{Z}^d \cap \square_m} \mathbb{E}\left[(\nu(z + \square_n, p) - \mathbb{E}[\nu(\square_n, p)])^2\right] \\
&= 3^{-d(m-n)} \times 3^d \operatorname{Var}[\nu(\square_n, p)] \\
&\leqslant C(d, \Lambda) 3^{-d(m-n)}
\end{aligned}$$

The second inequality comes from the fact that if $z + \square_n$ are not neighbors, they are independent. And the third inequality use the control of the variance of $\nu(\square_n, p)$, which is of course bounded by the eq. (3.1.2). □

Using Lemma 3.1.1, we obtain that

$$\mathbb{E}\left[\left|\nu(\square_m, p) - \frac{1}{2} p \cdot \bar{\mathbf{a}} p\right|\right] \leqslant C(d, \Lambda) 3^{-\frac{d}{2}(m-n)} + 2\omega(n) + \omega(m),$$

and we take $n = \lfloor \frac{m}{2} \rfloor$ to finish the first part of the proposition. □

The second part of Proposition 3.1.2 and Proposition 3.1.3 follow the same spirit, but sometimes some more techniques are used in the proof. We send reader to [AKM18, Proposition 1.4, Proposition 1.5] for their proof.

3.2 Estimate of a functional

The next goal may be to find a good estimate for the quantity $\omega(n)$. Instead of studying it directly, in [AKM18, Chapter 2], the authors define its dual quantity

$$\nu^*(U, q) = \sup_{v \in H^1(U)} \left(\frac{1}{|U|} \int_U -\frac{1}{2} \nabla v \cdot \mathbf{a} \nabla v + q \cdot \nabla v \right). \tag{3.2.1}$$

and propose to study another quantity that

$$J(U, p, q) := \nu(U, p) + \nu^*(U, q) - p \cdot q. \tag{3.2.2}$$

The motivation to do this transform is that by subtracting a dual quantity, the value $J(U, p, q)$ behaves like linear in large scale, which is the following theorem :

Theorem 3.2.1. *$\forall s \in (0,d), \exists \alpha(d,\Lambda) \in (0, \frac{1}{2})$ and $0 < C(s,d,\Lambda) < \infty$ such that*

$$\left| J(\square_n, p, q) - \left(\frac{1}{2} p \cdot \bar{\mathbf{a}} p + \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q - p \cdot q \right) \right| \leqslant C 3^{-n\alpha(d-s)} + \mathcal{O}_1(C 3^{-ns}).$$

With this analysis in hand, one can go back to the quantitative analysis of H-convergence and get their complete description.

3.3 Liouville regularity in large scale

The harmonic equation on the whole space has a nice regularity as described in Theorem 1.4.3, but it is not the case of a general for a general operator of divergence form. For example, in random walk context, we can construct a biased random walk that has probability $p \in (0,1)$ to return to the origin, and then the hitting probability defines a non-trivial harmonic function on whole space.

However, [AKM18, Chapter 3] states a remarkable result :

Theorem 3.3.1 (Liouville theorem for **a**-harmonic)**.** *We define*

$$\mathcal{A}_k = \left\{ u \in H^1_{loc}(\mathbb{R}^d), -\nabla \cdot \mathbf{a} \nabla u = 0 \text{ in } \mathbb{R}^d, \lim_{r \to \infty} r^{-(k+1)} \|u\|_{\underline{L}^2(B_r(0))} = 0 \right\}, \tag{3.3.1}$$

and $\overline{\mathcal{A}}_k$ as the similar definition by replacing $\mathbf{a}$ with $\bar{\mathbf{a}}$. Then, we have

$$dim(\mathcal{A}_k) = dim(\overline{\mathcal{A}}_k) = \binom{d+k-1}{k} + \binom{d+k-2}{k-1}.$$

Moreover, $\forall s \in (0,d), \exists \delta(s,d,\Lambda) \in \left(0, \frac{1}{2}\right]$ and a random variable $X_s \leqslant \mathcal{O}_s(C(s,d,\Lambda))$ allowing us to do a correspondence between $\mathcal{A}_k$ and $\overline{\mathcal{A}}_k$ for every $k \in \mathbb{N}$: $\exists C(k,d,\Lambda), \forall u \in \mathcal{A}_k, \exists p \in \overline{\mathcal{A}}_k$ and $\forall R \geqslant \mathcal{X}_s$ we have

$$\|u - p\|_{\underline{L}^2(B_R)} \leqslant C(k,d,\Lambda) R^{-\delta} \|p\|_{\underline{L}^2(B_R)}.$$

Reciprocally, $\forall p \in \overline{\mathcal{A}}_k, \exists u \in \mathcal{A}_k$ we can have the same estimate above.

That says in the large scale, the random operator $-\nabla \cdot \mathbf{a} \nabla$ improves the regularity. We remark that this result is later generalized in the context of the infinite cluster of percolation in [AD].

3.4 Two-scale expansion

The analysis above and the methods used in their proof allows us to give a complete quantitative description of the two-scale expansion in a stochastic coefficient context. Since we will prove a variant version in the following chapter, we do not repeat the result here. We send the readers to [AKM18, Chapter 6] and Section 4.3.

Chapter 4

Study of an iterative method

This chapter is adapted from one final project during my internship on an iterative algorithm solving the heterogeneous equation.

4.1 Introduction

4.1.1 Main theorem

The problem of homogenization is a subject widely studied in mathematics and other disciplines for its applications and interesting properties. We study the following elliptic equation where $\mathbf{a}$ stands for the coefficient of conductance and it is random, stationary defined with correlation distance 1 and uniformly elliptic in our context. Moreover, $r > 0, U_r = rU$ stands the area of the equation and usually the r is very big in concrete example. The source function is $f \in H^{-1}(U_r)$ and we try to find the solution in standard Sobolev space $u \in g + H_0^1(U_r) \subset H^1(U_r)$.

$$\begin{cases} -\nabla \cdot \mathbf{a}\nabla u = f & \text{in } U_r, \\ u = g & \text{on } \partial U_r. \end{cases} \tag{4.1.1}$$

However, the numerical algorithm for this problem is generally very expensive for the reason of the high oscillation on the coefficients. [AHKM18] proposes an iterative algorithm, which gives one approach to solve it quickly. The object of this article is to obtain a good uniform bound for this algorithm. We remark that both the algorithm and this article benefit a lot from a series of work [AS16], [AKM16], [AKM17] where we get a further progress in the context of stochastic homogenization and especially a quantitative estimate of the large scale regularity and first order corrector. Especially, [Mou16] designs an efficient method to calculate the effective matrix $\bar{\mathbf{a}}$, which provides us possibility to find new algorithms.

We give a quick introduction of the necessary notations and the main theorem. we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we denote $\mathcal{F}_V$ the σ-algebra generated by

$$\int_{\mathbb{R}^d} \chi \mathbf{a}_{i,j}, \text{ where } i,j \in \{1,2,3 \cdots d\}, \chi \in C_c^\infty(V).$$

$\mathcal{F}$ is short for $\mathcal{F}_{\mathbb{R}^d}$. T_y denotes an operator of translation i.e

$$T_y(a)(x) = \mathbf{a}(x+y).$$

In our context, we treat the stochastic homogenization problem where $\mathbf{a}$ represents a stationary random conductance of correlation distance 1 which satisfies the uniform elliptic condition.

1. *Stationarity* $\forall A \in \mathcal{F}$
$$\mathbb{P}[T_y A] = \mathbb{P}[A].$$

2. *Unit range correlation*
$$\forall U, V \in \mathcal{B}(\mathbb{R}^d), d_H(W, V) > 1 \iff \mathcal{F}_W, \mathcal{F}_V \text{ are independent.}$$
Here d_H is the Hausdorff distance in $\mathbb{R}^d$.

3. *Uniform ellipticity condition* $\forall x, \xi \in \mathbb{R}^d, \exists \Lambda > 0$ such that
$$\Lambda^{-1}|\xi|^2 \leqslant \xi \cdot \mathbf{a}(x)\xi \leqslant \Lambda|\xi|^2.$$

In the theory of homogenization, we use $\bar{\mathbf{a}}$ to represent the homogenized operator or effective conductance in large scale, which is a constant matrix. See [AKM18] for further details.

To calibrate the size of a random variable X, we use two parameters $s, \theta > 0$ and the notation $\mathcal{O}$ which is defined by
$$X \leqslant \mathcal{O}_s(1) \Longleftrightarrow \mathbb{E}\left[\exp((X \vee 0)^s)\right] \leqslant 2.$$
Informal by speaking, the statement $X \leqslant \mathcal{O}_s(1)$ tells us that X has a tail lighter than $\exp(-x^s)$. We note also that $X \leqslant \mathcal{O}_s(\theta)$ iff $\frac{X}{\theta} \leqslant \mathcal{O}_s(1)$. Our main theorem is that :

Theorem 4.1.1 (H^1 contraction)**.** *Given a domain U a bounded Borel set of $\mathbb{R}^d$ with $C^{1,1}$ boundary, for every $r \geqslant 2$, there exists an $\mathcal{F}$-measurable random variable $\mathcal{Z}$ satisfying, for every $s \in (0,2)$, there is a constant $0 < C(U, \Lambda, s, d) < \infty$, such that*
$$\mathcal{Z} \leqslant \mathcal{O}_s\left(C(U, \Lambda, s, d)(\log r)^{\frac{1}{s}}\right),$$
where $l(\lambda)$ is defined as
$$l(\lambda) = \begin{cases} (\log(1 + \lambda^{-1}))^{\frac{1}{2}} & d = 2, \\ 1 & d > 2. \end{cases} \tag{4.1.2}$$
and $\mathcal{Z}$ serves as a random factor of an iteration, such that the following holds:

Setting $U_r = rU$ and $\lambda \in (\frac{1}{r}, 1)$, for every $f \in H^{-1}(U_r), g \in H^1(U_r), v \in g + H^1_0(U_r)$ and $u \in g + H^1_0(U_r)$ the solution of eq. (4.1.1), let $u_0, \bar{u}, \widetilde{u} \in H^1_0(U)$ solve
$$\begin{cases} (\lambda^2 - \nabla \cdot \mathbf{a}\nabla)u_0 & = f + \nabla \cdot \mathbf{a}\nabla v & \text{in } U_r, \\ -\nabla \cdot \bar{\mathbf{a}}\nabla \bar{u} & = \lambda^2 u_0 & \text{in } U_r, \\ (\lambda^2 - \nabla \cdot \mathbf{a}\nabla)\widetilde{u} & = (\lambda^2 - \nabla \cdot \bar{\mathbf{a}}\nabla)\bar{u} & \text{in } U_r. \end{cases} \tag{4.1.3}$$
For $\hat{v} := v + u_0 + \widetilde{u}$, we have the concentration estimate
$$\|\hat{v} - u\|_{\underline{H}^1(U_r)} \leqslant l(\lambda)^{\frac{1}{2}} \lambda^{\frac{1}{2}} \mathcal{Z} \|v - u\|_{\underline{H}^1(U_r)}. \tag{4.1.4}$$

Thus, in this algorithm, the random factor $\mathcal{Z}$ is determined by the dimension, the random conductance $\mathbf{a}$ and the size of the area U_r. We can apply this algorithm for any suitable data and with a choice λ by ourselves. We start from an initial guess of solution v, for example, we take g as a trial and then iterate it. It suffices to take a small λ to reduce the effect of the typical constant of $\mathcal{Z}$, then we have large probability to have a contraction factor small than 1 and this will give us an exponential convergence since the contraction factor has a uniform bound for all data.

4.1.2 Heuristic analysis of algorithm

We give some very intuitive idea why this iteration can approximate the solution of eq. (4.1.1). Its philosophy is similar to the multi-scale algorithm. Each time we start with an initial guess of the solution v, then we write $u = v + (u - v)$ and we want to recover the part $(u - v)$. Since the divergence form is linear, we have

$$-\nabla \cdot \mathbf{a}\nabla(u - v) = -\nabla \cdot \mathbf{a}\nabla u + \nabla \cdot \mathbf{a}\nabla v = f + \nabla \cdot \mathbf{a}\nabla v,$$

so by the sense of weak solution, we have

$$(u - v) = \arg \min_{\varphi \in H_0^1(U_r)} \int_{U_r} \frac{1}{2}\nabla\varphi \cdot \mathbf{a}\varphi + \nabla v \cdot \mathbf{a}\varphi - f\varphi.$$

Instead of resolving this problem of minimization, we add some regularization to the problem and try to solve

$$u_0 = \arg \min_{\varphi \in H_0^1(U_r)} \int_{U_r} \frac{1}{2}\nabla\varphi \cdot \mathbf{a}\varphi + \frac{\lambda^2}{2}\varphi^2 + \nabla v \cdot \mathbf{a}\varphi - f\varphi.$$

This is exactly the first equation in the iteration and by the classical optimization theory, when λ goes to 0, u_0 converges to $(u-v)$ and in the regularized optimization problem, u_0 recovers the high oscillating part or the more detailed part of $(u - v)$. Informally, we write

$$(u - v) \approx (u - v)_{high} + (u - v)_{low},$$

such that

$$\begin{aligned}
\lambda^2 \int_{U_r} (u - v)^2_{high} &= o(1) \int_{U_r} \nabla(u - v)_{high} \cdot \mathbf{a}\nabla(u - v)_{high}, \\
\lambda^2 \int_{U_r} (u - v)^2_{low} &= O(1) \int_{U_r} \nabla(u - v)_{low} \cdot \mathbf{a}\nabla(u - v)_{low}.
\end{aligned}$$

Therefore, after the first iteration, we do not get all the information of $(u-v)$ but $u_0 \approx (u-v)_{high}$ and the second and third equation serve to recover $(u - v)_{low}$ of $(u - v)$. In fact, if we believe that $\widetilde{u} \approx (u - v - u_0) \approx (u - v)_{low}$ the direct idea is still to solve

$$-\nabla \cdot \mathbf{a}\nabla\widetilde{u} \approx -\nabla \cdot \mathbf{a}\nabla(u - v - u_0) = \lambda^2 u_0. \tag{4.1.5}$$

We want to solve a regularized problem instead of an original problem. That is, on the left hand side, we hope to appear $(\lambda^2 - \nabla \cdot \mathbf{a}\nabla)\widetilde{u}$. But this operation will have more or less effect on the right hand because eq. (4.1.5) tries to solve

$$\widetilde{u} \approx \arg \min_{\varphi \in H_0^1(U_r)} \int_{U_r} \frac{1}{2}\nabla\varphi \cdot \mathbf{a}\nabla\varphi - \lambda^2 u_0\varphi,$$

and it attains minimum by $\widetilde{u} \approx (u - v)_{low}$. So, adding one term of regularization means adding one term of energy $\int_{U_r} \frac{\lambda^2}{2}\varphi^2$. When $\varphi = \widetilde{u} \approx (u-v)_{low}$, this term cannot be neglected compared to $\int_{U_r} \frac{1}{2}\nabla\varphi \cdot \mathbf{a}\nabla\varphi$. For this reason, the corresponding optimization problem should be like

$$\widetilde{u} \approx \arg \min_{\varphi \in H_0^1(U_r)} \int_{U_r} \frac{1}{2}\nabla\varphi \cdot \mathbf{a}\nabla\varphi + \frac{\lambda^2}{2}\varphi^2 - \lambda^2 u_0\varphi - \lambda^2\widetilde{u}\varphi. \tag{4.1.6}$$

However, $\widetilde{u}$ cannot appear in the objective function to be minimized. That is why we introduce the second equation in the iteration. As a homogenized equation,

$$-\nabla \cdot \bar{\mathbf{a}}\nabla \bar{u} = -\nabla \cdot \mathbf{a}\nabla(u - v - u_0) = \lambda^2 u_0, \tag{4.1.7}$$

implies that in $\frac{1}{|U_r|}\|\bar{u} - (u - v - u_0)\|_{L^2(U_r)} \xrightarrow{r\to 0} 0$ in classical homogenization theory, see for example [JKO12]. So in L^2 weak sense, we have $\bar{u} \approx (u - v - u_0) \approx \widetilde{u}$, then we replace $\widetilde{u}$ by $\bar{u}$ in eq. (4.1.6) to get

$$\widetilde{u} = \arg \min_{\varphi \in H_0^1(U_r)} \int_{U_r} \frac{1}{2}\nabla\varphi \cdot \mathbf{a}\nabla\varphi + \frac{\lambda^2}{2}\varphi^2 - \lambda^2 u_0\varphi - \lambda^2\bar{u}\varphi.$$

Then it turns out to be the third equation in eq. (4.1.1)

$$(\lambda^2 - \nabla \cdot \mathbf{a}\nabla)\widetilde{u} = \lambda^2(u_0 + \bar{u}) = (\lambda^2 - \nabla \cdot \bar{\mathbf{a}}\nabla)\bar{u}. \tag{4.1.8}$$

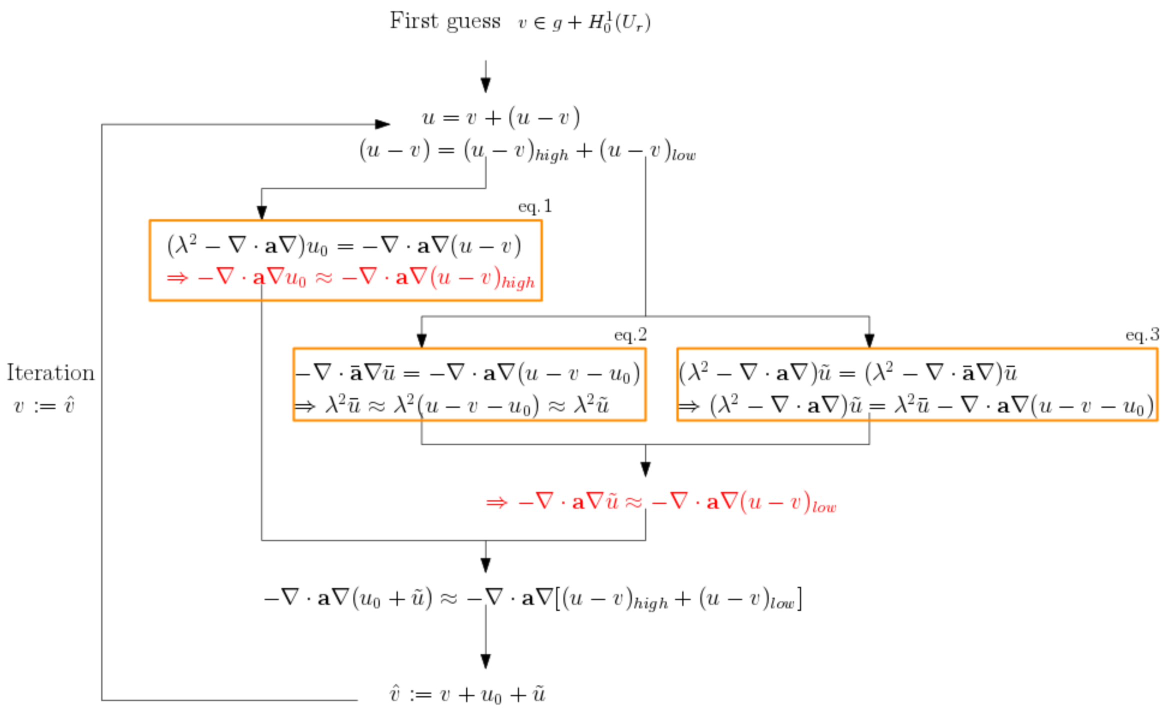

Figure 4.1: A flowchart shows the mechanic of the algorithm

Finally, we add some remarks on the choice of λ. We insist solving regularized equation in stead of the original one for the efficiency in numeric and, generally speaking, a big λ gives better stability in numeric schema. On the other hand, we know the algorithm converges when the factor is less than 1, which implies a choice of small λ so that it happens in high probability. So we have to do good equilibrium between a smaller contraction factor and a better numerical stability. Here, one typical recommendation of choice is $1 > \lambda \gg \frac{1}{r}$.

4.1.3 Connection with previous work

The strategy and main structure used in this article come from [AHKM18] and [AKM18, Chapter 6], some improvement is also inspired by the classical work of two-scale expansion in periodic context. The following part talks about the main idea of the proof and the improvement compared to the previous work.

At the very beginning, we recall the two-scale expansion, a very useful concept in the theory of homogenization. It first appears in a similar problem as eq. (4.1.1) but in periodic context: Let $\mathbf{a}$ stands the conductance matrix with coefficient of period ε and $\bar{\mathbf{a}}$ its homogenized matrix and they determine a set of functions $\{\phi_e\}_{e\in\mathbb{R}^d}$ called *(first order) corrector* for every $e \in \mathbb{R}^d$ and for every $g \in H^1(U)$ and $u^\varepsilon, \bar{u} \in g + H^1_0(U)$ such that

$$-\nabla \cdot \mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) \nabla u^\varepsilon = -\nabla \cdot \bar{\mathbf{a}} \nabla \bar{u}, \tag{4.1.9}$$

we can use *two-scale expansion (of first order)*, namely a linear combination of $\bar{u}$ and corrector associated to d canonical unit vectors e_k

$$w^\varepsilon := \bar{u} + \varepsilon \sum_{k=1}^{d} \partial_{x_k} \bar{u}\, \phi_{e_k}\left(\frac{\cdot}{\varepsilon}\right), \tag{4.1.10}$$

to approximate the solution u^ε in the sense H^1

$$\|w^\varepsilon - u^\varepsilon\|_{H^1(U)} \leqslant O(\sqrt{\varepsilon}). \tag{4.1.11}$$

See [All92] for detailed introduction. This is a very nice idea since in periodic context, the corrector can be computed directly by $\mathbf{a}$ without other knowledge. Then, eq. (4.1.10) gives us a quick numerical solution, that is to use the two-scale expansion as an approximated solution of the equation. See for example [HW97] and [AB05].

In random conductance context, there are similar ideas as two-scale expansion, but to transform them to a numerical algorithm is difficult for two reasons : Firstly, different from deterministic situation, the corrector cannot be computed explicitly. Thus, some other pre-treatment like oversampling should be applied if we hope to obtain eq. (4.1.10) Secondly, a quantitative description of corrector like eq. (4.1.11) in random conductance context is absent for long time.

Thanks to the recent progress in quantitative description of the corrector in stochastic homogenization, we now solve the second difficulty mentioned above. See [AKM18, Chapter 6] for its proof. This also gives us possibility to calibrate some numerical algorithms in stochastic context. In [AHKM18], the algorithm in Theorem 4.1.1 is proposed for the first time under the same condition, but it states its result as

$$\|\hat{v} - u\|_{\underline{H}^1(U_r)} \leqslant \mathcal{O}_s\left(C(U,\Lambda,s,d) l^{\frac{1}{2}}(\lambda)\lambda^{\frac{1}{2}} \|v - u\|_{\underline{H}^1(U_r)}\right). \tag{4.1.12}$$

and it uses the two-scale expansion as a tool rather than to approximate directly the solution of eq. (4.1.1). Since the proof in this artilce follows generally the similar main steps to that in [AHKM18]. We give the plot to readers.

1. *Step 1 : Prove a modified two-scale expansion theorem.* We hope to work in one equation similar to eq. (4.1.9) but with one term of regularization like

   $$\left(\mu^2 - \nabla \cdot \mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\right) \nabla u^\varepsilon = \left(\mu^2 - \nabla \cdot \bar{\mathbf{a}} \nabla\right) \bar{u}. \tag{4.1.13}$$

   Its proof contains two parts and the first part is like in periodic situation which is deterministic, and it corresponds to the first two subsections of Section 4.3.

2. *Step 2 : Do some quantitative analysis of corrector.* This part aims to conclude the result from the first step and randomness comes to the analysis. It corresponds to the last two subsections of Section 4.3.

3. *Step 3 : Apply the two-scale expansion lemma in algorithm.* From Section 4.1.2, we get two equations eq. (4.1.7), eq. (4.1.8) which are two combinations from eq. (4.1.3). They satisfy exactly the condition of eq. (4.1.13). So we do

$$\|(v + u_0 + \widetilde{u}) - u\|_{\underline{H}^1(U_r)} \leqslant \|w - (u - v - u_0)\|_{\underline{H}^1(U_r)} + \|\widetilde{u} - w\|_{\underline{H}^1(U_r)}, \tag{4.1.14}$$

and apply the two-scale expansion theorem twice on the right hand side. This part will be done in Section 4.4.

Finally, we state the contribution and improvement in this article. If we check the result in eq. (4.1.4) and eq. (4.1.12), we see that the result in former improves in the sense :

- Bound in eq. (4.1.4) is a uniform while the bound in eq. (4.1.12) depends on data. The price is that we add a small factor of $(\log r)^{\frac{1}{s}}$.

This improvement is necessary for this algorithm : if the bound varies with respect with the function, since the function also changes, we do not know if the algorithm really converges. The worst situation can be that the algorithm has bad chance and always goes to function with even larger bound. In order to separate the random factor $\mathcal{Z}$ aside, we have to do precise estimate in *Step 2*. More precisely, one technical task is to estimate a term like $\|(\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}})\partial_{x_k}\bar{u}\|_{\underline{L}^2(U_r)}$ where $(\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}})$ is random and $\bar{u}$ is deterministic. We want to show

$$\|(\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}})\partial_{x_k}\bar{u}\|_{\underline{L}^2(U_r)} \leqslant \|\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}}\|_{L^\infty(U_r)}\|\partial_{x_k}\bar{u}\|_{\underline{L}^2(U_r)}.$$

But we only have the point-wise estimate and their linear combination of $(\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}})$ on hand rather than the type of L^∞ norm. So, we use a technique of localization from [AKM18, Chapter 6] to change the right hand side in a discrete version and then we handle it. Section 4.2 is devoted to this technical estimate and we prove a new Lemma 4.2.2, which did not appear in the previous references. We remark that for the reason to apply these new technique, the definition of two-scale expansion theorem 4.3.1 is a little different from that in [AHKM18].

4.1.4 Organization of paper

In the rest of the article, we give the proof of theorem 4.1.1. In section 4.2, we give two lemmas that improve or generalize the technique of [AKM18] in our context. The rest is to reformulate our technique in the structure of [AHKM18]. In section 4.3, we prove a theorem of two-scale expansion which will be heavily used in the later part. Finally, in section 4.4, we combine all the results and obtain the main theorem.

4.2 Two technical lemmas

In this section, we prove two useful lemmas that will be used in later work. A formula similar to lemma 4.2.1 can be found in [AKM18, Lemme 6.7]. Here we introduce a variant version and it works well together with lemma 4.2.2.

4.2.1 An inequality of localization

Lemma 4.2.1 (Mixed norm)**.** *There exists a constant $0 < C(d) < \infty$ such that for every $g, f \in L^2(\mathbb{R}^d)$ and every $\varepsilon > 0, r \geqslant 2$, we have the following inequality*

$$\|f(g \star \zeta_\varepsilon)\|_{\underline{L}^2(U_r)} \leqslant C(d)\left(\max_{z\in\varepsilon\mathbb{Z}^d\cap U_r}\|f\|_{\underline{L}^2(z+\varepsilon\square_0)}\right)\|g\|_{\underline{L}^2(U_r+2\varepsilon\square_0)}. \tag{4.2.1}$$

Proof.

$$\begin{aligned}
\|f(g\star\zeta_\varepsilon)\|^2_{\underline{L}^2(U_r)} &= \frac{1}{|U_r|}\sum_{z\in\varepsilon\mathbb{Z}^d\cap U_r}\|f(g\star\zeta_\varepsilon)\|^2_{L^2(z+\varepsilon\square_0)} \\
\text{(Hölder's inequality)} \quad &\leqslant \frac{1}{|U_r|}\sum_{z\in\varepsilon\mathbb{Z}^d\cap U_r}\left(\|f\|^2_{L^2(z+\varepsilon\square_0)}\|g\star\zeta_\varepsilon\|^2_{L^\infty(z+\varepsilon\square_0)}\right) \\
&\leqslant \frac{1}{|U_r|}\left(\max_{z\in\varepsilon\mathbb{Z}^d\cap U_r}\|f\|^2_{L^2(z+\varepsilon\square_0)}\right)\left(\sum_{z\in\varepsilon\mathbb{Z}^d\cap U_r}\|g\star\zeta_\varepsilon\|^2_{L^\infty(z+\varepsilon\square_0)}\right).
\end{aligned}$$

Noticing that $\forall x\in z+\varepsilon\square_0$,

$$\begin{aligned}
|g\star\zeta_\varepsilon(x)| &= \left|\int_{\varepsilon\square_0} g(x-y)\frac{1}{\varepsilon^d}\zeta(\frac{y}{\varepsilon})\,dy\right| \\
&\leqslant \frac{C(d)}{\varepsilon^d}\int_{z+2\varepsilon\square_0}|g(y)|\,dy \\
&\leqslant \frac{C(d)}{\varepsilon^d}\left(\int_{z+2\varepsilon\square_0}|g(y)|^2\,dy\right)^{\frac{1}{2}}|2\varepsilon\square_0|^{\frac{1}{2}} \\
&\leqslant \frac{C(d)}{\varepsilon^{\frac{d}{2}}}\|g\|_{L^2(z+2\varepsilon\square_0)}.
\end{aligned}$$

So we get

$$\|g\star\zeta_\varepsilon\|_{L^\infty(z+\varepsilon\square_0)}\leqslant\frac{C(d)}{\varepsilon^{\frac{d}{2}}}\|g\|_{L^2(z+2\varepsilon\square_0)},$$

and we add this analysis in the former inequality and obtain that

$$\begin{aligned}
\|f(g\star\zeta_\varepsilon)\|^2_{\underline{L}^2(U_r)} &\leqslant C(d)\left(\frac{1}{\varepsilon^d}\max_{z\in\varepsilon\mathbb{Z}^d\cap U_r}\|f\|^2_{L^2(z+\varepsilon\square_0)}\right)\left(\frac{1}{|U_r|}\sum_{z\in\varepsilon\mathbb{Z}^d\cap U_r}\|g\|^2_{L^2(z+2\varepsilon\square_0)}\right) \\
&= C(d)\left(\max_{z\in\varepsilon\mathbb{Z}^d\cap U_r}\|f\|^2_{\underline{L}^2(z+\varepsilon\square_0)}\right)\|g\|^2_{\underline{L}^2(U_r+2\varepsilon\square_0)}.
\end{aligned}$$

This is the desired inequality. □

4.2.2 Maximum of finite number of random variables of type $\mathcal{O}_s(1)$

Since $\left(\max_{z\in\varepsilon\mathbb{Z}^d\cap U_r}\|f\|_{\underline{L}^2(z+\varepsilon\square_0)}\right)$ often appears in the next paragraph as the maximum of a family of random variables, we prepare a lemma to analyze the maximum of a finite number of random variables of type $\mathcal{O}_s(1)$. Note that we do not make any assumptions on the joint law of the random variables.

Lemma 4.2.2. *For all $N\geqslant 1$ and a family of random variables $\{X_i\}_{1\leqslant i\leqslant N}$ satisfying that $X_i\leqslant\mathcal{O}_s(1)$, we have*

$$\left(\max_{1\leqslant i\leqslant N}X_i\right)\leqslant\mathcal{O}_s\left(\left(\frac{\log(2N)}{\log(3/2)}\right)^{\frac{1}{s}}\right). \tag{4.2.2}$$

Proof. By the Markov inequality,

$$X_i\leqslant\mathcal{O}_s(1)\Longrightarrow\mathbb{P}[X_i>x]\leqslant 2e^{-x^s}.$$

By a union bound, we get

$$\begin{aligned}\mathbb{P}\left[\max_{1\leqslant i\leqslant N} X_i > x\right] &= \mathbb{P}\left[\bigcup_{i=1}^{N}\{X_i > x\}\right] \\ &\leqslant \left(1 \wedge \sum_{i=1}^{N}\mathbb{P}[X_i > x]\right) \\ &\leqslant 1 \wedge 2Ne^{-x^s}.\end{aligned}$$

We denote by x_0 the critical point such that $e^{x_0^s} = 2N$ and we set $M = \max_{1\leqslant i\leqslant N} X_i$ and $a > 0$ such that $a^s > 2$ which will be chosen carefully later. Then we use the Fubini formula

$$\begin{aligned}\mathbb{E}\left[\exp\left(\left(\frac{M}{a}\right)_+^s\right)\right] &= \int_0^\infty \frac{s}{a}\left(\frac{x}{a}\right)^{s-1} e^{(\frac{x}{a})^s}\mathbb{P}[M > x]\,dx \\ &= \int_0^{x_0} \frac{s}{a}\left(\frac{x}{a}\right)^{s-1} e^{(\frac{x}{a})^s}\underbrace{\mathbb{P}[M > x]}_{\leqslant 1}\,dx + \int_{x_0}^\infty \frac{s}{a}\left(\frac{x}{a}\right)^{s-1} e^{(\frac{x}{a})^s}\underbrace{\mathbb{P}[M > x]}_{\leqslant 2Ne^{-x^s}}\,dx \\ &\leqslant \int_0^{x_0} \frac{s}{a}\left(\frac{x}{a}\right)^{s-1} e^{(\frac{x}{a})^s}\,dx + 2N\int_{x_0}^\infty \frac{s}{a}\left(\frac{x}{a}\right)^{s-1} e^{(\frac{x}{a})^s - x^s}\,dx \\ &= \int_0^{x_0} e^{(\frac{x}{a})^s}\,d\left(\frac{x}{a}\right)^s + 2N\int_{x_0}^\infty e^{(\frac{x}{a})^s - x^s}\,d\left(\frac{x}{a}\right)^s \\ &= \int_0^{(\frac{x_0}{a})^s} e^y\,dy + 2N\int_{(\frac{x_0}{a})^s}^\infty e^{-y(a^s-1)}\,dy \\ &= \left(e^{(\frac{x_0}{a})^s} - 1\right) + \frac{2N}{a^s - 1}e^{-(\frac{x_0}{a})^s(a^s-1)} \\ &= (2N)^{\frac{1}{a^s}} - 1 + \frac{1}{a^s-1}(2N)^{\frac{1}{a^s}} \\ &\leqslant 2(2N)^{\frac{1}{a^s}} - 1 \\ &= 2e^{\frac{\log(2N)}{a^s}} - 1.\end{aligned}$$

Now we fix $a = \left(\frac{\log(2N)}{\log(3/2)}\right)^{\frac{1}{s}}$, for the case $N \geqslant 2$, we have

$$a^s = \left(\frac{\log(2N)}{\log(3/2)}\right) \geqslant \left(\frac{\log(4)}{\log(3/2)}\right) > \left(\frac{\log(4)}{\log(2)}\right) = 2,$$

so by the definition of $\mathcal{O}_s$

$$\mathbb{E}\left[\exp\left(\left(\frac{M}{a}\right)_+^s\right)\right] \leqslant 2e^{\log(3/2)} - 1 = 2.$$

For the case $N = 1$, we could check that eq. (4.2.2) is also established since

$$M = X_1 \leqslant \mathcal{O}_s(1) \Longrightarrow M \leqslant \mathcal{O}_s\left(\frac{\log(4)}{\log(3/2)}\right).$$

This finishes the proof. □

4.3 A modified two-scale expansion estimate

4.3.1 Main structure

One important inspiration of this algorithm may be the two-scale expansion, which says that we can approximate the solution of eq. (4.1.1) by the solution of the homogenized equation and the *first order corrector* ϕ_e. At first, we give its definition.

Definition 4.3.1 (First order corrector)**.** For each $e \in \mathbb{R}^d$, the corrector ϕ_e is the sublinear function satisfying that $e \cdot x + \phi_e$ is **a**-harmonic in whole space $\mathbb{R}^d$ i.e

$$-\nabla \cdot \mathbf{a}(e + \nabla \phi_e) = 0, \text{ in } \mathbb{R}^d. \tag{4.3.1}$$

ϕ_e is defined up to a constant.

The properties of the correctors are recently developed in [AKM18]. We just remark that the set correctors forms a vector space that we can associate every d canonical unit vector e_k a first order corrector ϕ_{e_k}. The reader can also find the quantitative description of the two-scale expansion in [AKM18, Chapter 6]. The idea here is to use a $(\lambda^2 - \nabla \cdot a\nabla)$ version of two-scale expansion to evaluate the difference between the solution of the iteration and the real one.

Theorem 4.3.1 (Two-scale estimate)**.** *There exists a constant* $0 < C(U, \Lambda, d) < \infty$*,* $\mathcal{F}$*-measurable random variables* $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$*, and for each* $s \in (0, 2)$*, there exists a constant* $C'(U, s, d)$ *such that for every* $\lambda \in (\frac{1}{r}, \frac{1}{2}], r \geqslant 2$ *and* $\bar{v} \in H_0^1(U_r) \cap H^2(U_r)$*, we define*

$$\begin{aligned} \phi_e^{(\lambda)} &:= \phi_e - \phi_e \star \Phi_{\lambda^{-1}} \\ w &:= \bar{v} + \sum_{k=1}^{d} \partial_{x_k}(\bar{v} \star \zeta)\phi_{e_k}^{(\lambda)}, \end{aligned}$$

which is a two-scale expansion. For every $\mu \in [0, \lambda]$ *and* $v \in H_0^1(U)$ *such that*

$$\begin{cases} (\mu^2 - \nabla \cdot a\nabla)v = (\mu^2 - \nabla \cdot \bar{\mathbf{a}}\nabla)\bar{v} & \textit{in } U, \\ v = \bar{v} & \textit{on } \partial U, \end{cases} \tag{4.3.2}$$

we have the H^1 *estimate*

$$\boxed{\begin{aligned} \|v - w\|_{\underline{H}^1(U_r)} \leqslant C(U, \Lambda, d) \Big[&\|\bar{v}\|_{\underline{H}^2(U_r)} + \left(\|\bar{v}\|_{\underline{H}^2(U_r)} + \mu\|\bar{v}\|_{\underline{H}^1(U_r)}\right)\mathcal{X}_1 \\ &+ \left(l(\lambda)^{\frac{1}{2}}\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^2(U_r)}\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^1(U_r)} + \|\bar{v}\|_{\underline{H}^1(U_r)}\right)\mathcal{X}_2 \\ &+ \left(l(\lambda)^{\frac{1}{2}}\left(\mu + \frac{1}{r} + \frac{1}{l(\lambda)}\right)\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^2(U_r)}\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^1(U_r)} + \|\bar{v}\|_{\underline{H}^2(U_r)}\right)\mathcal{Y}_1 \Big], \end{aligned}}$$

and $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$ *satisfy that*

$$\mathcal{X}_1 \leqslant \mathcal{O}_s\left(C'(U, s, d)l(\lambda)(\log r)^{\frac{1}{s}}\right), \qquad \mathcal{X}_2 \leqslant \mathcal{O}_s\left(C'(U, s, d)\lambda^{\frac{d}{2}}(\log r)^{\frac{1}{s}}\right), \tag{4.3.4}$$

$$\mathcal{Y}_1 \leqslant \mathcal{O}_s\left(C'(U, s, d)l(\lambda)(\log r)^{\frac{1}{s}}\right). \tag{4.3.5}$$

Remark. The explicit expression of $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$ will be given later in the proof. They could be seen as the maximum of local spatial average of gradient and flux of first order corrector.

Proof. We give at first the proof of deterministic part. We will see that the errors can finally be reduced to the estimates of two norms : the interior error term and a boundary layer term. The latter boundary layer term comes to the fact that v and w do not have the same boundary condition. So we propose b the solution of the equation

$$\begin{cases} (\mu^2 - \nabla \cdot \mathbf{a}\nabla) b = 0 & \text{in } U_r, \\ b(x) = \sum_{k=1}^{d} \partial_{x_k}(\bar{v} \star \zeta)\phi_{e_k}^{(\lambda)} & \text{on } \partial U_r. \end{cases} \tag{4.3.6}$$

Then $w - b$ shares the same boundary condition as v. So, we have

$$\|v - w\|_{\underline{H}^1(U_r)} \leqslant \|v + b - w\|_{\underline{H}^1(U_r)} + \|b\|_{\underline{H}^1(U_r)}, \tag{4.3.7}$$

and we do estimates of the two parts respectively.

- **Estimate for z.** We denote $z := v + b - w \in H_0^1(U_r)$, that is why we regard it as the source of error in interior part. Since $z \in H_0^1(U_r)$, we test it in eq. (4.3.2) and eq. (4.3.6)

$$\begin{aligned} \mu^2 \int_{U_r} zv + \int_{U_r} \nabla z \cdot \mathbf{a}\nabla v &= \mu^2 \int_{U_r} z\bar{v} + \int_{U_r} \nabla z \cdot \bar{\mathbf{a}}\nabla \bar{v} \\ \mu^2 \int_{U_r} zb + \int_{U_r} \nabla z \cdot \mathbf{a}\nabla b &= 0. \end{aligned}$$

We do the sum to obtain that

$$\mu^2 \int_{U_r} z(v+b) + \int_{U_r} \nabla z \cdot \mathbf{a}\nabla(v+b) = \mu^2 \int_{U_r} z\bar{v} + \int_{U_r} \nabla z \cdot \bar{\mathbf{a}}\nabla\bar{v}.$$

Using the fact $v + b = z + w$, we obtain

$$\mu^2 \int_{U_r} |z|^2 + \int_{U_r} \nabla z \cdot \mathbf{a}\nabla z = \mu^2 \int_{U_r} z(\bar{v} - w) + \int_{U_r} \nabla z \cdot (\bar{\mathbf{a}}\nabla\bar{v} - \mathbf{a}\nabla w).$$

and we apply the uniform ellipticity condition to obtain

$$\begin{aligned} \mu^2 \|z\|^2_{\underline{L}^2(U_r)} + \Lambda^{-1}\|\nabla z\|^2_{\underline{L}^2(U_r)} &\leqslant \mu^2 \|z\|_{\underline{L}^2(U_r)} \|w - \bar{v}\|_{\underline{L}^2(U_r)} \\ &\quad + \|z\|_{\underline{H}^1(U_r)} \|\nabla \cdot \mathbf{a}\nabla w - \nabla \cdot \bar{\mathbf{a}}\nabla\bar{v}\|_{\underline{H}^{-1}(U_r)} \\ \text{(Young's inequality)} &\leqslant \mu^2 \|z\|^2_{\underline{L}^2(U_r)} + \frac{\mu^2}{4}\|w - \bar{v}\|^2_{\underline{L}^2(U_r)} \\ &\quad + \frac{\Lambda^{-1}}{2}\|z\|^2_{\underline{H}^1(U_r)} + \frac{\Lambda}{2}\|\nabla \cdot \mathbf{a}\nabla w - \nabla \cdot \bar{\mathbf{a}}\nabla\bar{v}\|^2_{\underline{H}^{-1}(U_r)} \\ \Longrightarrow \|\nabla z\|^2_{\underline{L}^2(U_r)} &\leqslant \Lambda \|\nabla \cdot \mathbf{a}\nabla w - \nabla \cdot \bar{\mathbf{a}}\nabla\bar{v}\|_{\underline{H}^{-1}(U_r)} + \sqrt{\Lambda}\mu\|w - \bar{v}\|_{\underline{L}^2(U_r)}. \end{aligned}$$

We use Poincaré's inequality to conclude that

$$\|z\|^2_{\underline{H}^1(U_r)} \leqslant C(U)\left(\Lambda \|\nabla \cdot \mathbf{a}\nabla w - \nabla \cdot \bar{\mathbf{a}}\nabla\bar{v}\|_{\underline{H}^{-1}(U_r)} + \sqrt{\Lambda}\mu\|w - \bar{v}\|_{\underline{L}^2(U_r)}\right) \tag{4.3.8}$$

- **Estimate for b.** To estimate b we use the property that it is the optimizer of the problem

$$J = \inf_{\chi \in b + H_0^1(U_r)} \mu^2 \int_{U_r} \chi^2 \, dx + \int_{U_r} \nabla\chi \cdot a\nabla\chi \, dx.$$

So we give an upper bound of this functional by a sub-optimizer

$$T_\lambda := \left(\mathbf{1}_{\mathbb{R}^d \setminus U_{r,2l(\lambda)}} \star \zeta_{l(\lambda)}\right) \sum_{k=1}^{d} \partial_{x_k}(\bar{v} \star \zeta)\phi_{e_k}^{(\lambda)},$$

where $U_{r,2l(\lambda)}$ is defined as

$$U_{r,2l(\lambda)} = \{x \in U_r | d(x, \partial U_r) > 2l(\lambda)\}.$$

The motivation to propose this sub-optimizer is the following : If we think the solution of elliptic equation is an average in some sense of the boundary value, then when the coefficient is oscillating, the boundary value is hard to propagate. So one naive candidate is just smoothing the boundary value in a small band of length $2l(\lambda)$.

By comparison,

$$\begin{aligned} \mu^2 \int_{U_r} |b|^2 + \int_{U_r} \nabla b \cdot \mathbf{a}\nabla b &\leqslant \mu^2 \int_{U_r} |T_\lambda|^2 + \int_{U_r} \nabla T_\lambda \cdot \mathbf{a}\nabla T_\lambda \\ \Longrightarrow \|\nabla b\|_{\underline{L}^2(U_r)} &\leqslant \mu\sqrt{\Lambda}\|T_\lambda\|_{\underline{L}^2(U_r)} + \Lambda\|\nabla T_\lambda\|_{\underline{L}^2(U_r)}. \end{aligned}$$

Moreover, to estimate the L^2 norm, we use once again the Poincaré's inequality

$$\begin{aligned} \|b\|_{\underline{L}^2(U_r)} &= \|b - T_\lambda + T_\lambda\|_{\underline{L}^2(U_r)} \\ &\leqslant \|b - T_\lambda\|_{\underline{L}^2(U_r)} + \|T_\lambda\|_{\underline{L}^2(U_r)} \\ \text{(Poincaré's inequality)} \quad &\leqslant r\|\nabla(b - T_\lambda)\|_{\underline{L}^2(U_r)} + \|T_\lambda\|_{\underline{L}^2(U_r)} \\ &\leqslant r\|\nabla b\|_{\underline{L}^2(U_r)} + r\|\nabla T_\lambda\|_{\underline{L}^2(U_r)} + \|T_\lambda\|_{\underline{L}^2(U_r)}. \end{aligned}$$

We combine the two and get an estimate of b

$$\begin{aligned} \|b\|_{\underline{H}^1(U_r)} &= \frac{1}{|U_r|^{\frac{1}{d}}}\|b\|_{\underline{L}^2(U_r)} + \|\nabla b\|_{\underline{L}^2(U_r)} \\ &\leqslant C(U)\left(\frac{1}{r}\|T_\lambda\|_{\underline{L}^2(U_r)} + \mu\sqrt{\Lambda}\|T_\lambda\|_{\underline{L}^2(U_r)} + (1+\Lambda)\|\nabla T_\lambda\|_{\underline{L}^2(U_r)}\right). \end{aligned}$$

Finally, we put all the estimates above into eq. (4.3.7)

$$\|v - w\|_{\underline{H}^1(U_r)} \leqslant C(U,\Lambda)\Bigg(\|\mathbf{a}\nabla w - \bar{\mathbf{a}}\nabla\bar{v}\|_{\underline{H}^{-1}(U_r)} + \mu\|w - \bar{v}\|_{\underline{L}^2(U_r)} \tag{4.3.9a}$$

$$+\|\nabla T_\lambda\|_{\underline{L}^2(U_r)} + (\frac{1}{r} + \mu)\|T_\lambda\|_{\underline{L}^2(U_r)}\Bigg). \tag{4.3.9b}$$

To complete the proof of theorem 4.3.1, we have to treat these random norms respectively. It is the main task of the next section. □

4.3.2 Construction of a flux corrector

A very useful technique in the analysis of $\|\mathbf{a}\nabla w - \bar{\mathbf{a}}\nabla\bar{v}\|_{\underline{H}^{-1}(U_r)}$ is to construct a flux corrector. Similar formulas appear both in [AHKM18] and [AKM18], here we give the version in our context.

For every $e \in \mathbb{R}^d$, since $\mathbf{g}_e := \mathbf{a}(e + \nabla\phi_e) - \bar{\mathbf{a}}e)$ defines a divergence free field, i.e. $\nabla \cdot \mathbf{g}_e = 0$, it admits a representation as the "curl" of some potential vector by Helmholtz's theorem. That is $\mathbf{S}_e$, which is a skew-symmetric matrix $\mathbf{S}_e$ such that

$$\mathbf{a}(e + \nabla\phi_e) - \bar{\mathbf{a}}e = \nabla \cdot \mathbf{S}_e,$$

where $\nabla \cdot \mathbf{S}_e$ is a $\mathbb{R}^d$ valued vector defined by $(\nabla \cdot \mathbf{S}_e)_i = \sum_{j=1}^d \partial_{x_j} \mathbf{S}_{e,ij}$ In order to "fix the gauge", for each $i, j \in \{1, 2, \cdots d\}$, we force

$$\Delta \mathbf{S}_{e,ij} = \partial_{x_j} \mathbf{g}_{e,i} - \partial_{x_i} \mathbf{g}_{e,j},$$

and under this condition, $\mathbf{S}_e$ is unique up to a constant. We set also

$$\mathbf{S}^{(\lambda)}_{e_k} = \mathbf{S}_{e_k} - \mathbf{S}_{e_k} \star \Phi_{\lambda^{-1}}.$$

We have the following identity.

Lemma 4.3.1. *For $\lambda > 0, \bar{v} \in H^1(U_r)$ and $w \in H^1(U_r)$ as in theorem 4.3.1. We construct a vector field $\mathbf{F}$ such that*

$$\nabla \cdot (\mathbf{a}\nabla w - \bar{\mathbf{a}}\nabla\bar{v}) = \nabla \cdot \mathbf{F},$$

whose i-th component is given by

$$\begin{aligned} \mathbf{F}_i &= \sum_{j=1}^d (\mathbf{a}_{ij} - \bar{\mathbf{a}}_{ij}) \partial_{x_j}(\bar{v} - \bar{v} \star \zeta) + \sum_{j,k=1}^d \left(\mathbf{a}_{ij}\phi^{(\lambda)}_{e_k} - \mathbf{S}^{(\lambda)}_{e_k,ij} \right) \partial_{x_j}\partial_{x_k}(\bar{v} \star \zeta) \\ &\quad + \sum_{j,k=1}^d \left(\partial_{x_j} \mathbf{S}_{e_k,ij} \star \Phi_{\lambda^{-1}} - \mathbf{a}_{ij}\partial_{x_j}\phi_{e_k} \star \Phi_{\lambda^{-1}} \right) \partial_{x_k}(\bar{v} \star \zeta). \end{aligned}$$

Proof. We develop

$$\begin{aligned} [\mathbf{a}\nabla w - \bar{\mathbf{a}}\nabla\bar{v}]_i &= \left[\mathbf{a}\nabla \left(\bar{v} + \sum_{k=1}^d \partial_{x_k}(\bar{v} \star \zeta)\phi^{(\lambda)}_{e_k} \right) - \bar{\mathbf{a}}\nabla\bar{v} \right]_i \\ &= \left[(\mathbf{a} - \bar{\mathbf{a}})\nabla\bar{v} + \nabla \sum_{k=1}^d \left(\partial_{x_k}(\bar{v} \star \zeta)\phi^{(\lambda)}_{e_k} \right) \right]_i \\ &= \underbrace{[(\mathbf{a} - \bar{\mathbf{a}})\nabla(\bar{v} - \bar{v} \star \zeta)]_i}_{\mathbf{I}} + \underbrace{\left[(\mathbf{a} - \bar{\mathbf{a}})\nabla(\bar{v} \star \zeta) + \mathbf{a}\nabla \sum_{k=1}^d \left(\partial_{x_k}(\bar{v} \star \zeta)\phi^{(\lambda)}_{e_k} \right) \right]_i}_{\mathbf{II}}. \end{aligned}$$

The first term is indeed

$$\mathbf{I} = [(\mathbf{a} - \bar{\mathbf{a}})\nabla(\bar{v} - \bar{v} \star \zeta)]_i = \sum_{j=1}^d (\mathbf{a}_{ij} - \bar{\mathbf{a}}_{ij}) \partial_{x_j}(\bar{v} - \bar{v} \star \zeta),$$

as in the right hand side of the identity, so we continue to study the rest of the formula.

$$\begin{aligned}
\mathbf{II} &= \sum_{j=1}^{d}(\mathbf{a}_{ij} - \bar{\mathbf{a}}_{ij})\partial_{x_j}(\bar{v} \star \zeta) + \sum_{j,k=1}^{d} \mathbf{a}_{ij}\partial_{x_j}\left(\partial_{x_k}(\bar{v} \star \zeta)\phi_{e_k}^{(\lambda)}\right) \\
&= \sum_{j,k=1}^{d} (\mathbf{a}_{ij} - \bar{\mathbf{a}}_{ij})\partial_{x_j}(\bar{v} \star \zeta)\delta_{jk} + \sum_{j,k=1}^{d} \mathbf{a}_{ij}\partial_{x_j}\partial_{x_k}(\bar{v} \star \zeta)\phi_{e_k}^{(\lambda)} + \sum_{j,k=1}^{d} \mathbf{a}_{ij}\partial_{x_k}(\bar{v} \star \zeta)\partial_{x_j}\phi_{e_k}^{(\lambda)} \\
&= \underbrace{\sum_{j,k=1}^{d} \left((\mathbf{a}_{ij} - \bar{\mathbf{a}}_{ij})\delta_{jk} + \mathbf{a}_{ij}\partial_{x_j}\phi_{e_k}^{(\lambda)}\right)\partial_{x_k}(\bar{v} \star \zeta)}_{\mathbf{II.1}} + \underbrace{\sum_{j,k=1}^{d} \mathbf{a}_{ij}\partial_{x_j}\partial_{x_k}(\bar{v} \star \zeta)\phi_{e_k}^{(\lambda)}}_{\mathbf{II.2}}.
\end{aligned}$$

II.2 appears in the right hand side of the formula, so it remains **II.1** to treat. We use the the definition of $\mathbf{S}_{e_k}^{(\lambda)}$ in **II.1**

$$\begin{aligned}
\mathbf{II.1} &= \sum_{k=1}^{d} \left[\mathbf{a}(e_k + \nabla\phi_{e_k}^{(\lambda)}) - \bar{\mathbf{a}}e_k\right]_i \partial_{x_k}(\bar{v} \star \zeta) \\
&= \sum_{k=1}^{d} \left[\mathbf{a}(e_k + \nabla\phi_{e_k}) - \bar{\mathbf{a}}e_k - \mathbf{a}\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}}\right]_i \partial_{x_k}(\bar{v} \star \zeta) \\
&= \sum_{k=1}^{d} \left[\nabla \cdot \mathbf{S}_{e_k} - \mathbf{a}\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}}\right]_i \partial_{x_k}(\bar{v} \star \zeta) \\
&= \underbrace{\sum_{k=1}^{d} \left[\nabla \cdot \mathbf{S}_{e_k}^{(\lambda)}\right]_i \partial_{x_k}(\bar{v} \star \zeta)}_{\mathbf{III}} + \sum_{k=1}^{d} \left[\nabla \cdot \mathbf{S}_{e_k} \star \Phi_{\lambda^{-1}} - \mathbf{a}\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}}\right]_i \partial_{x_k}(\bar{v} \star \zeta).
\end{aligned}$$

All the terms match well except **III**, where we have to look for an equal form after divergence. Thanks to the property of skew-symmetry, we have

$$\begin{aligned}
\nabla \cdot \mathbf{III} &= \nabla \cdot \left(\sum_{k=1}^{d} \left[\nabla \cdot \mathbf{S}_{e_k}^{(\lambda)}\right]_i \partial_{x_k}(\bar{v} \star \zeta)\right) \\
&= \sum_{i,j,k=1}^{d} \partial_{x_i}\left(\partial_{x_j}\mathbf{S}_{e_k,ij}^{(\lambda)}\,\partial_{x_k}(\bar{v} \star \zeta)\right) \\
\text{(Integration by parts)} &= \sum_{i,j,k=1}^{d} \partial_{x_i}\partial_{x_j}\left(\mathbf{S}_{e_k,ij}^{(\lambda)}\partial_{x_k}(\bar{v} \star \zeta)\right) - \partial_{x_i}\left(\mathbf{S}_{e_k,ij}^{(\lambda)}\,\partial_{x_j}\partial_{x_k}(\bar{v} \star \zeta)\right) \\
\text{(Skew-symmetry of } \mathbf{S}\text{)} &= -\nabla \cdot \left(\sum_{j,k=1}^{d} \mathbf{S}_{e_k,ij}^{(\lambda)}\,\partial_{x_j}\partial_{x_k}(\bar{v} \star \zeta)\right).
\end{aligned}$$

This finishes the proof. □

4.3.3 Quantitative description of $\phi_{e_k}^{(\lambda)}$ and $\mathbf{S}_{e_k}^{(\lambda)}$

In this subsection, we will give some quantitative description of $\phi_{e_k}^{(\lambda)}$ and $\mathbf{S}_{e_k}^{(\lambda)}$, which could be seen as the spatial average of the corrector in cell and serves as the bricks to form $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$.

Lemma 4.3.2 (Estimate of corrector). *For each $s \in (0,2)$ there exists a constant $0 < C(s,\Lambda,d) < \infty$ such that for every $\lambda \in (0,1)$, $i,j,k \in \{1,\cdots d\}, z \in \mathbb{Z}^d$*

$$\|\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)} \leqslant \mathcal{O}_s(C\lambda^{\frac{d}{2}}), \qquad \|\nabla \mathbf{S}_{e_k,ij} \star \Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)} \leqslant \mathcal{O}_s(C\lambda^{\frac{d}{2}}),$$
$$\|\phi^{(\lambda)}_{e_k}\|_{\underline{L}^2(z+\square_0)} \leqslant \mathcal{O}_s(Cl(\lambda)), \qquad \|\mathbf{S}^{(\lambda)}_{e_k,ij}\|_{\underline{L}^2(z+\square_0)} \leqslant \mathcal{O}_s(Cl(\lambda)).$$

Proof. We talk only about the part $\phi^{(\lambda)}_{e_k}$. [AKM18, Theorem 4.1] gives us three useful estimates

- $d \geqslant 2, r > 1$,
$$|\nabla\phi_{e_k} \star \Phi_r| \leqslant \mathcal{O}_s(C(s,d,\Lambda)r^{-\frac{d}{2}}) \tag{4.3.10}$$
- $d \geqslant 3$,
$$\|\phi_{e_k}\|_{\underline{L}^2(\square_0)} \leqslant \mathcal{O}_2(C(d,\Lambda)) \tag{4.3.11}$$
- $d = 2, 2 \leqslant r < R < \infty, x, y \in \mathbb{R}^d$,
$$\|\phi_{e_k} - \phi_{e_k} \star \Phi_r(0)\|_{\underline{L}^2(r\square_0)} \leqslant \mathcal{O}_s(C(s,\Lambda)\log^{\frac{1}{2}} r) \tag{4.3.12a}$$
$$|(\phi_{e_k} * \Phi_r)(x) - (\phi_{e_k} * \Phi_R)(y)| \leqslant \mathcal{O}_s\left(C(s,\Lambda)\log^{\frac{1}{2}}\left(2 + \frac{R + |x-y|}{r}\right)\right). \tag{4.3.12b}$$

and moreover when $d \geqslant 3$ there exists a $\mathbb{Z}^d$ stationary ϕ_{e_k} by choice $\mathbb{E}\left[\int_{\square_0} \phi_{e_k}\right] = 0$.

1. Proof of $\|\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)} \leqslant \mathcal{O}_s(C\lambda^{\frac{d}{2}})$.

   The first inequality implies $\|\phi^{(\lambda)}_{e_k}\|_{\underline{L}^2(z+\square_0)} \leqslant \mathcal{O}_s(Cl^{\frac{1}{2}}(\lambda))$: by choosing $r = \lambda^{-1}$ and using eq. (1.5.2), we have

$$\begin{aligned} |\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}}(x)| &\leqslant \mathcal{O}_s(C\lambda^{\frac{d}{2}}) \\ \Longrightarrow |\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}}(x)|^2 &\leqslant \mathcal{O}_{s/2}(C^2\lambda^d) \\ \Longrightarrow \int_{z+\square_0} |\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}}(x)|^2\,dx &\leqslant \mathcal{O}_{s/2}(C^2\lambda^d) \\ \Longrightarrow \|\nabla\phi_{e_k} \star \Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)} &\leqslant \mathcal{O}_s(C\lambda^{\frac{d}{2}}). \end{aligned}$$

2. Proof that if $d \geqslant 3$, then $\|\phi^{(\lambda)}_{e_k}\|_{\underline{L}^2(z+\square_0)} \leqslant \mathcal{O}_s(C)$. We apply the second inequality and use the stationarity of ϕ to get that

$$\|\phi^{(\lambda)}_{e_k}\|_{\underline{L}^2(z+\square_0)} \leqslant \|\phi_{e_k}\|_{\underline{L}^2(z+\square_0)} + \|\phi_{e_k} \star \Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)},$$

   where the first one has been controlled. In fact, by the stationarity we have $\|\phi_{e_k}\|_{\underline{L}^2(z+\square_0)} \leqslant \mathcal{O}_2(C(d,\Lambda))$, this means for every $s \in (0,2)$,

$$\begin{aligned} \mathbb{E}\left[\exp\left(\left(\frac{\|\phi_{e_k}\|_{\underline{L}^2(z+\square_0)}}{C(d,\Lambda)}\right)^s_+\right)\right] &\leqslant \mathbb{E}\left[\exp\left(\left(\frac{\|\phi_{e_k}\|_{\underline{L}^2(z+\square_0)}}{C(d,\Lambda)}\right)^2_+\right)\right] \leqslant 2 \\ \Longrightarrow \|\phi_{e_k}\|_{\underline{L}^2(z+\square_0)} &\leqslant \mathcal{O}_s(C(d,\Lambda)). \end{aligned}$$

Thus we focus on the second one that

$$\begin{aligned}
\|\phi_{e_k} \star \Phi_{\lambda^{-1}}\|^2_{\underline{L}^2(z+\square_0)} &= \int_{z+\square_0} \left| \int_{\mathbb{R}^d} \phi_{e_k}(x-y)\Phi_{\lambda^{-1}}(y)\,dy \right|^2 dx \\
&= \int_{z+\square_0} \left| \int_{\mathbb{R}^d} \phi_{e_k}(x-y)\Phi^{\frac{1}{2}}_{\lambda^{-1}}(y)\Phi^{\frac{1}{2}}_{\lambda^{-1}}(y)\,dy \right|^2 dx \\
\text{(Hölder's inequality)} \quad &\leqslant \int_{z+\square_0} \left(\int_{\mathbb{R}^d} \phi^2_{e_k}(x-y)\Phi_{\lambda^{-1}}(y)\,dy \right) \underbrace{\left(\int_{\mathbb{R}^d} \Phi_{\lambda^{-1}}(y)\,dy \right)}_{=1} dx \\
&= \int_{z+\square_0} \int_{\mathbb{R}^d} \phi^2_{e_k}(x-y)\Phi_{\lambda^{-1}}(y)\,dy\,dx \\
(eq.\ (1.5.2)) \quad &\leqslant \mathcal{O}_s(C).
\end{aligned}$$

In the last step, we treat $\Phi_{\lambda^{-1}}$ as a weight for different small cubes so we could apply eq. (1.5.2) and the stationarity of ϕ_{e_k}.

3. Proof that if $d = 2$, then $\|\phi^{(\lambda)}_{e_k}\|_{\underline{L}^2(z+\square_0)} \leqslant \mathcal{O}_s(Cl(\lambda))$.

   This part is a little more difficult than the case $d \geqslant 3$ since the scale of integration is different with the scale of convolution. So we need a intermediate step.

   Lemma 4.3.3. *There exist a constant* $0 < C(s, d, \Lambda) < \infty$ *such that*

$$\forall R \geqslant 1, \quad |\phi_{e_k} \star \Phi_R(x) - \phi_{e_k} \star \Phi_R(y)| \leqslant \mathcal{O}_s(C|x-y|R^{-1}). \tag{4.3.13}$$

   Proof.

$$\begin{aligned}
& |\phi_{e_k} \star \Phi_R(x) - \phi_{e_k} \star \Phi_R(y)| \\
= & \left| \int_0^1 \nabla\phi_{e_k} \star \Phi_R(x + t(y-x)) \cdot (y-x)\,dt \right| \\
\leqslant & |y-x| \int_0^1 |\nabla\phi_{e_k} \star \Phi_R(x + t(y-x))|\,dt \\
\leqslant & \mathcal{O}_s(C|y-x|R^{-1}).
\end{aligned}$$

   The last step combines the estimate of $\nabla\phi_{e_k} \star \Phi_R$ and eq. (1.5.2).

   □

   We apply eq. (4.3.13) eq. (4.3.12a) eq. (4.3.12b) to $\|\phi^{(\lambda)}_{e_k}\|_{\underline{L}^2(z+\square_0)}$ for $\forall\lambda \in (0, \frac{1}{2}]$

$$\begin{aligned}
\|\phi^{(\lambda)}_{e_k}\|_{\underline{L}^2(z+\square_0)} &= \|\phi_{e_k} - \phi_{e_k} \star \Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)} \\
&\leqslant \|\phi_{e_k} - \phi_{e_k} \star \Phi_2(z)\|_{\underline{L}^2(z+\square_0)} + \|\phi_{e_k} \star \Phi_2(z) - \phi_{e_k} \star \Phi_2\|_{\underline{L}^2(z+\square_0)} \\
&\quad + \|\phi_{e_k} \star \Phi_2 - \phi_{e_k} \star \Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)} \\
&\leqslant 4\underbrace{\|\phi_{e_k} - \phi_{e_k} \star \Phi_2(z)\|_{\underline{L}^2(B_2(z))}}_{\text{Apply } eq.\ (4.3.12a)} + 4\underbrace{\|\phi_{e_k} \star \Phi_2(z) - \phi_{e_k} \star \Phi_2\|_{\underline{L}^2(B_2(z))}}_{\text{Apply } eq.\ (4.3.13)} \\
&\quad + \underbrace{\|\phi_{e_k} \star \Phi_2 - \phi_{e_k} \star \Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)}}_{\text{Apply } eq.\ (4.3.12b)} \\
&\leqslant \mathcal{O}_s(C) + \mathcal{O}_s(C) + \mathcal{O}_s(C \log^{\frac{1}{2}}(2 + (2\lambda)^{-1})) \\
(eq.\ (1.5.2)) \quad &\leqslant \mathcal{O}_s(Cl(\lambda)).
\end{aligned}$$

Here we use Φ_2 since eq. (4.3.12a) eq. (4.3.12b) requires that the scale should be bigger than 2. In last step, we use also the condition $\lambda \leqslant \frac{1}{2}$ to give up the constant term.

Since $\mathbf{S}_{e_k}$ has the same type of estimate as ϕ_{e_k}, see [AKM18, Proposition 6.2] , we apply the same procedure to obtain the other half of the lemma 4.3.2. □

4.3.4 Detailed H^{-1} and boundary layer estimate

In this subection, we complete the proof of theorem 4.3.1, which remains to give an explicit random variable in the formula eq. (4.3.9a). This requires to analyze several norms like $\|a\nabla w - \bar{\mathbf{a}}\nabla\bar{v}\|_{\underline{H}^{-1}(U_r)}, \|w-\bar{v}\|_{\underline{L}^2(U_r)}$ where $\|a\nabla w - \bar{\mathbf{a}}\nabla\bar{v}\|_{\underline{H}^{-1}(U_r)}$ has an equivalent expression from last section. We will use a variant version of the localization technique in chapiter 6 of [AKM18] to separate the random factor and then use two technical lemmas in section 4.2 to calibrate the size of the random factor.

Estimate of $\|\mathbf{a}\nabla w - \bar{\mathbf{a}}\nabla\bar{v}\|_{\underline{H}^{-1}(U_r)}$

With the help of lemma 4.3.1, we have

$$\|\mathbf{a}\nabla w - \bar{\mathbf{a}}\nabla\bar{v}\|_{\underline{H}^{-1}(U_r)} = \|\nabla\cdot\mathbf{F}\|_{\underline{H}^{-1}(U_r)} \leqslant \|\mathbf{F}\|_{\underline{L}^2(U_r)},$$

and we use the identity lemma 4.3.1 to obtain

$$\begin{aligned}\|\mathbf{F}\|_{\underline{L}^2(U_r)} \quad \leqslant \quad & \underbrace{\sum_{j=1}^{d}\|(\mathbf{a}-\bar{\mathbf{a}})\nabla(\bar{v}-\bar{v}\star\zeta)\|_{\underline{L}^2(U_r)}}_{\mathbf{H.1}} \\ & + \underbrace{\sum_{j,k=1}^{d}\left\|\left(\mathbf{a}\phi_{e_k}^{(\lambda)} - \mathbf{S}_{e_k}^{(\lambda)}\right)\partial_{x_j}\partial_{x_k}(\bar{v}\star\zeta)\right\|_{\underline{L}^2(U_r)}}_{\mathbf{H.2}} \\ & + \underbrace{\sum_{k=1}^{d}\|(\nabla\mathbf{S}_{e_k}\star\Phi_{\lambda^{-1}} - \mathbf{a}\nabla\phi_{e_k}\star\Phi_{\lambda^{-1}})\,\partial_{x_k}(\bar{v}\star\zeta)\|_{\underline{L}^2(U_r)}}_{\mathbf{H.3}}.\end{aligned}$$

We treat the three terms respectively. For $\mathbf{H.1}$, we have

$$\mathbf{H.1} \leqslant d\Lambda\|\nabla\bar{v} - \nabla\bar{v}\star\zeta\|_{\underline{L}^2(U_r)} \leqslant d\Lambda\|\bar{v}\|_{\underline{H}^2(U_r)},$$

where the last step comes from the approximation of identity, see for example [AKM18, Lemma 6.7].

For $\mathbf{H.2}$, since $\|\phi_{e_k}^{(\lambda)}\|_{\underline{L}^2(z+\square_0)}, \|\mathbf{S}_{e_k}^{(\lambda)}\|_{\underline{L}^2(z+\square_0)}$ are obtained in lemma 4.3.2, we could use the lemma 4.2.1 where we treat the cell of the scale $\varepsilon = 1$ and take $g = \partial_{x_j}\partial_{x_k}\bar{v}, f = (\mathbf{a}\phi_{e_k}^{(\lambda)} - \mathbf{S}_{e_k}^{(\lambda)})$

$$\begin{aligned}\mathbf{H.2} \quad &= \quad \sum_{j,k=1}^{d}\|\left(\mathbf{a}\phi_{e_k}^{(\lambda)} - \mathbf{S}_{e_k}^{(\lambda)}\right)\partial_{x_j}\partial_{x_k}(\bar{v}\star\zeta)\|_{\underline{L}^2(U_r)} \\ (\textit{lemma } 4.2.1) \quad &\leqslant \quad C(\Lambda,d)\sum_{k=1}^{d}\max_{z\in\mathbb{Z}^d\cap U_r}\left(\|\phi_{e_k}^{(\lambda)}\|_{\underline{L}^2(z+\square_0)} + \|\mathbf{S}_{e_k}^{(\lambda)}\|_{\underline{L}^2(z+\square_0)}\right)\|\bar{v}\|_{\underline{H}^2(U_r)}.\end{aligned}$$

Hence, we extract the term of random variable

$$\mathcal{X}_1 := \sum_{k=1}^{d} \max_{z \in \mathbb{Z}^d \cap U_r} \left(\|\phi_{e_k}^{(\lambda)}\|_{\underline{L}^2(z+\square_0)} + \|\mathbf{S}_{e_k}^{(\lambda)}\|_{\underline{L}^2(z+\square_0)} \right), \tag{4.3.14}$$

lemma 4.2.2 and lemma 4.3.2 can be applied here to calibrate the size of random variables that

$$\mathcal{X}_1 \leqslant \mathcal{O}_s \left(C(U,s,d) l(\lambda) (\log r)^{\frac{1}{s}} \right).$$

The above estimation gives a good recipe for the remaining part. For **H.3**, we have

$$\begin{aligned} \mathbf{H.3} &= \sum_{k=1}^{d} \| \left(\nabla \mathbf{S}_{e_k} \star \Phi_{\lambda^{-1}} - a \nabla \phi_{e_k} \star \Phi_{\lambda^{-1}} \right) \partial_{x_k} (\bar{v} \star \zeta) \|_{\underline{L}^2(U_r)} \\ &\leqslant C(\Lambda, d) \sum_{k=1}^{d} \max_{z \in \mathbb{Z}^d \cap U_r} \left(\|\nabla \phi_{e_k} \star \Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)} + \|\nabla \mathbf{S}_{e_k} \star \Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)} \right) \|\bar{v}\|_{\underline{H}^1(U_r)}, \end{aligned}$$

where we extract that

$$\mathcal{X}_2 := \sum_{k=1}^{d} \max_{z \in \mathbb{Z}^d \cap U_r} \left(\|\nabla \phi_{e_k} \star \Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)} + \|\nabla \mathbf{S}_{e_k} \star \Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)} \right), \tag{4.3.15}$$

and we apply lemma 4.2.2 and lemma 4.3.2 to get

$$\mathcal{X}_2 \leqslant \mathcal{O}_s \left(C(U,s,d) \lambda^{\frac{d}{2}} (\log r)^{\frac{1}{s}} \right).$$

Combing **H.1**, **H.2**, **H.3**, we get

$$\boxed{\|\mathbf{a}\nabla w - \bar{\mathbf{a}}\nabla \bar{v}\|_{\underline{H}^{-1}(U_r)} \leqslant C(\Lambda, d) \left(\|\bar{v}\|_{\underline{H}^2(U_r)} + \|\bar{v}\|_{\underline{H}^2(U_r)} \mathcal{X}_1, + \|\bar{v}\|_{\underline{H}^1(U_r)} \mathcal{X}_2 \right)}. \tag{4.3.16}$$

Estimate of $\|w - \bar{v}\|_{\underline{L}^2(U_r)}$

For $\|w - \bar{v}\|_{\underline{L}^2(U_r)}$, it's just routine and

$$\begin{aligned} \|w - \bar{v}\|_{\underline{L}^2(U_r)} &= \| \sum_{k=1}^{d} \phi_{e_k}^{(\lambda)} \partial_{x_k} \bar{v} \star \zeta \|_{\underline{L}^2(U_r)} \\ (\textit{lemma } 4.2.1) \quad &\leqslant C(\Lambda, d) \sum_{k=1}^{d} \max_{z \in \mathbb{Z}^d \cap U_r} \left(\|\phi_{e_k}^{(\lambda)}\|_{\underline{L}^2(z+\square_0)} \right) \|\bar{v}\|_{\underline{H}^1(U_r)} \\ &\leqslant C(\Lambda, d) \|\bar{v}\|_{\underline{H}^1(U_r)} \mathcal{X}_1. \end{aligned}$$

$$\Longrightarrow \boxed{\|w - \bar{v}\|_{\underline{L}^2(U_r)} \leqslant C(\Lambda, d) \|\bar{v}\|_{\underline{H}^1(U_r)} \mathcal{X}_1}. \tag{4.3.17}$$

Estimate of $\|\nabla T_\lambda\|_{\underline{L}^2(U_r)}, \|T_\lambda\|_{\underline{L}^2(U_r)}$

Finally, we come to the estimate of $\|\nabla T_\lambda\|_{\underline{L}^2(U_r)}, \|T_\lambda\|_{\underline{L}^2(U_r)}$. We study $\|\nabla T_\lambda\|_{\underline{L}^2(U_r)}$ at first.

$$\begin{aligned}\|\nabla T_\lambda\|_{\underline{L}^2(U_r)} &= \underbrace{\left\|\left(\mathbf{1}_{U_r\setminus U_{r,2l(\lambda)}} \star \frac{1}{l^{\frac{d}{2}+1}(\lambda)}(\nabla\zeta)\left(\frac{\cdot}{l(\lambda)}\right)\right)\sum_{k=1}^{d}\partial_{x_k}(\bar{v}\star\zeta)\phi^{(\lambda)}_{e_k}\right\|_{\underline{L}^2(U_r)}}_{\mathbf{T.1}} \\ &+ \underbrace{\left\|\left(\mathbf{1}_{U_r\setminus U_{r,2l(\lambda)}} \star \zeta_{l(\lambda)}\right)\sum_{k=1}^{d}\partial_{x_k}(\nabla\bar{v}\star\zeta)\phi^{(\lambda)}_{e_k}\right\|_{\underline{L}^2(U_r)}}_{\mathbf{T.2}} \\ &+ \underbrace{\left\|\left(\mathbf{1}_{U_r\setminus U_{r,2l(\lambda)}} \star \zeta_{l(\lambda)}\right)\sum_{k=1}^{d}\partial_{x_k}(\bar{v}\star\zeta)\nabla\phi^{(\lambda)}_{e_k}\right\|_{\underline{L}^2(U_r)}}_{\mathbf{T.3}}.\end{aligned}$$

$$\begin{aligned}\mathbf{T.1} &\leqslant C\frac{1}{l(\lambda)}\left\|\sum_{k=1}^{d}\mathbf{1}_{U_r\setminus U_{r,2l(\lambda)}}\partial_{x_k}(\bar{v}\star\zeta)\phi^{(\lambda)}_{e_k}\right\|_{\underline{L}^2(U_r)} \\ &\leqslant \frac{C(\Lambda,d)}{l(\lambda)}\sum_{k=1}^{d}\max_{z\in\mathbb{Z}^d\cap U_r\setminus U_{r,2l(\lambda)}}\left(\|\phi^{(\lambda)}_{e_k}\|_{\underline{L}^2(z+\square_0)}\right)\|\mathbf{1}_{U_r\setminus U_{r,2l(\lambda)}}\bar{v}\|_{\underline{H}^1(U_r)}.\end{aligned}$$

We see that lemma 4.2.1 and lemma 4.2.2 also work, but we should pay attention to one small improvement : the domain of integration is in fact restricted in $U_r\setminus U_{r,2l(\lambda)}$, so we would like to give it a bound in terms of $\underline{H}^2(U_r)$ rather than $aH^1(U_r)$. We borrow a trace estimate in [AHKM18] Proposition A.1 that for $f\in H^1(U_r)$

$$\|f\mathbf{1}_{U_r\setminus U_{r,2l(\lambda)}}\|_{\underline{L}^2(U_r)} \leqslant C(U,d)l(\lambda)^{\frac{1}{2}}\|f\|^{\frac{1}{2}}_{\underline{H}^1(U_r)}\|f\|^{\frac{1}{2}}_{\underline{L}^2(U_r)}, \tag{4.3.18}$$

using eq. (4.3.18) then we obtain an estimate

$$\mathbf{T.1} \leqslant \frac{C(\Lambda,d)}{l^{\frac{1}{2}}(\lambda)}\sum_{k=1}^{d}\max_{z\in\mathbb{Z}^d\cap U_r\setminus U_{r,2l(\lambda)}}\left(\|\phi^{(\lambda)}_{e_k}\|_{\underline{L}^2(z+\square_0)}\right)\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^2(U_r)}\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^1(U_r)},$$

so we define the random variable

$$\mathcal{Y}_1 := \sum_{k=1}^{d}\max_{z\in\mathbb{Z}^d\cap(U_r\setminus U_{r,2l(\lambda)})}\left(\|\phi^{(\lambda)}_{e_k}\|_{\underline{L}^2(z+\square_0)}\right), \tag{4.3.19}$$

and we have the estimate by lemma 4.2.2 and lemma 4.3.2

$$\mathcal{Y}_1 \leqslant \mathcal{O}_s\left(C(U,s,d)l(\lambda)(\log r)^{\frac{1}{s}}\right).$$

We skip the details since they are analogue to the previous part. **T.3** follows from the same type of estimate as **T.1** and **T.2** is routine where we suffices to apply lemma 4.2.1 and eq. (4.3.18). We find that

$$\begin{aligned}\mathbf{T.2} &\leqslant C(U,d)\|\bar{v}\|_{\underline{H}^2(U_r)}\mathcal{Y}_1, \\ \mathbf{T.3} &\leqslant C(U,d)\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^2(U_r)}\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^1(U_r)}l^{\frac{1}{2}}(\lambda)\mathcal{X}_2.\end{aligned}$$

The three estimates of $\mathbf{T.1}, \mathbf{T.2}, \mathbf{T.3}$ implies that

$$\boxed{\begin{aligned}\|\nabla T_\lambda\|_{\underline{L}^2(U_r)} &\leqslant C(\Lambda,d)\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^2(U_r)}\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^1(U_r)}\frac{1}{l^{\frac{1}{2}}(\lambda)}\mathcal{Y}_1 && (4.3.20a)\\ &+ C(U,d)\left(\|\bar{v}\|_{\underline{H}^2(U_r)}\mathcal{Y}_1 + \|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^2(U_r)}\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^1(U_r)}l^{\frac{1}{2}}(\lambda)\mathcal{X}_2\right) && (4.3.20b)\end{aligned}}$$

Finally, we find that $\|T_\lambda\|_{\underline{L}^2(U_r)}$ has been contained in the estimate $\mathbf{T.1}$ that

$$\boxed{\|T_\lambda\|_{\underline{L}^2(U_r)} \leqslant C(U,d)\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^2(U_r)}\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^1(U_r)}l^{\frac{1}{2}}(\lambda)\mathcal{Y}_1}. \tag{4.3.21}$$

eq. (4.3.9a), eq. (4.3.16), eq. (4.3.17), eq. (4.3.20), eq. (4.3.21) conclude the proof of theorem 4.3.1. We have

$$\begin{aligned}\|v-w\|_{\underline{H}^1(U_r)} &\leqslant C(U,\Lambda)\Bigg(\|\mathbf{a}\nabla w - \bar{\mathbf{a}}\nabla\bar{v}\|_{\underline{H}^{-1}(U_r)} + \mu\|w-\bar{v}\|_{\underline{L}^2(U_r)}\\ &\quad +\|\nabla T_\lambda\|_{\underline{L}^2(U_r)} + (\frac{1}{r}+\mu)\|T_\lambda\|_{\underline{L}^2(U_r)}\Bigg)\\ &\leqslant C(U,\Lambda,d)\Bigg(\|\bar{v}\|_{\underline{H}^2(U_r)} + \|\bar{v}\|_{\underline{H}^2(U_r)}\mathcal{X}_1 + \|\bar{v}\|_{\underline{H}^1(U_r)}\mathcal{X}_2 + \mu\|\bar{v}\|_{\underline{H}^1(U_r)}\mathcal{X}_1\\ &\quad +l(\lambda)^{\frac{1}{2}}\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^2(U_r)}\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^1(U_r)}\left(\left(\mu+\frac{1}{r}+\frac{1}{l(\lambda)}\right)\mathcal{Y}_1+\mathcal{X}_2\right) + \|\bar{v}\|_{\underline{H}^2(U_r)}\mathcal{Y}_1\Bigg)\\ &= C(U,\Lambda,d)\Bigg[\|\bar{v}\|_{\underline{H}^2(U_r)} + \left(\|\bar{v}\|_{\underline{H}^2(U_r)} + \mu\|\bar{v}\|_{\underline{H}^1(U_r)}\right)\mathcal{X}_1\\ &\quad + \left(l(\lambda)^{\frac{1}{2}}\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^2(U_r)}\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^1(U_r)} + \|\bar{v}\|_{\underline{H}^1(U_r)}\right)\mathcal{X}_2\\ &\quad + \left(l(\lambda)^{\frac{1}{2}}\left(\mu+\frac{1}{r}+\frac{1}{l(\lambda)}\right)\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^2(U_r)}\|\bar{v}\|^{\frac{1}{2}}_{\underline{H}^1(U_r)} + \|\bar{v}\|_{\underline{H}^2(U_r)}\right)\mathcal{Y}_1\Bigg]\end{aligned}$$

We add one table of $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$ to check the its typical size..

R.V	Expression	$\mathcal{O}_s$ size
$\mathcal{X}_1$	$\sum_{k=1}^d \max_{z\in\mathbb{Z}^d\cap U_r}\left(\|\phi^{(\lambda)}_{e_k}\|_{\underline{L}^2(z+\square_0)} + \|\mathbf{S}^{(\lambda)}_{e_k}\|_{\underline{L}^2(z+\square_0)}\right)$	$\mathcal{O}_s\left(Cl(\lambda)(\log r)^{\frac{1}{s}}\right)$
$\mathcal{X}_2$	$\sum_{k=1}^d \max_{z\in\mathbb{Z}^d\cap U_r}\left(\|\nabla\phi_{e_k}\star\Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)} + \|\nabla\mathbf{S}_{e_k}\star\Phi_{\lambda^{-1}}\|_{\underline{L}^2(z+\square_0)}\right)$	$\mathcal{O}_s\left(C\lambda^{\frac{d}{2}}(\log r)^{\frac{1}{s}}\right)$
$\mathcal{Y}_1$	$\sum_{k=1}^d \max_{z\in\mathbb{Z}^d\cap(U_r\setminus U_{r,2l(\lambda)})}\left(\|\phi^{(\lambda)}_{e_k}\|_{\underline{L}^2(z+\square_0)}\right)$	$\mathcal{O}_s\left(Cl(\lambda)(\log r)^{\frac{1}{s}}\right)$

Figure 4.2: A table of random variables $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$

4.4 Iteration estimate

In this part, we use theorem 4.3.1 to analyze the algorithm, but first we give an H^1, H^2 à priori estimate to explain why we do some detailed analysis in the part of trace : the H^2 norm allows us to gain one factor of λ in compared to the H^1 norm.

4.4.1 Proof of a H^1, H^2 estimate

Lemma 4.4.1. *In eq. (4.1.3), we have a control*

$$\|\bar{u}\|_{\underline{H}^1(U_r)} + \lambda^{-1}\|\bar{u}\|_{\underline{H}^2(U_r)} \leqslant C(U,\Lambda,d)\|v-u\|_{\underline{H}^1(U_r)}.$$

Proof. We test the first equation $(\lambda^2 - \nabla\cdot\mathbf{a}\nabla)u_0 = -\nabla\cdot\mathbf{a}\nabla(u-v)$ in eq. (4.1.3) by u_0 and use the ellipticity condition to obtain

$$\begin{aligned}
\lambda^2\|u_0\|^2_{L^2(U_r)} + \Lambda^{-1}\|\nabla u_0\|^2_{L^2(U_r)} &\leqslant \lambda^2\|u_0\|^2_{L^2(U_r)} + \int_{U_r}\nabla u_0\cdot\mathbf{a}\nabla u_0\,dx \\
&= \int_{U_r}\nabla u_0\cdot\mathbf{a}\nabla(u-v)\,dx \\
&\leqslant \Lambda\|\nabla(v-u)\|_{L^2(U_r)}\|\nabla u_0\|_{L^2(U_r)} \\
\Longrightarrow \|\nabla u_0\|_{\underline{L}^2(U_r)} &\leqslant \Lambda\|\nabla(v-u)\|_{\underline{L}^2(U_r)}.
\end{aligned}$$

We put back this term in the inequality, we also obtain that

$$\lambda\|u_0\|_{\underline{L}^2(U_r)} \leqslant \Lambda\|\nabla(v-u)\|_{\underline{L}^2(U_r)}. \tag{4.4.1}$$

Using this estimate, we obtain that of $\nabla\bar{u}$ by testing $-\nabla\cdot\bar{\mathbf{a}}\nabla\bar{u} = -\nabla\cdot\mathbf{a}\nabla(u-v-u_0)$ with $\bar{u}$

$$\begin{aligned}
\int_{U_r}\nabla\bar{u}\cdot\bar{\mathbf{a}}\nabla\bar{u}\,dx &= \int_{U_r}\nabla\bar{u}\cdot\mathbf{a}\nabla(u-v-u_0)\,dx \\
\Longrightarrow \|\nabla\bar{u}\|_{\underline{L}^2(U_r)} &\leqslant \Lambda^2\|\nabla(u-v-u_0)\|_{\underline{L}^2(U_r)} \\
&\leqslant \Lambda^2\|\nabla(u-v)\|_{\underline{L}^2(U_r)} + \Lambda^2\|\nabla u_0\|_{\underline{L}^2(U_r)} \\
&\leqslant C(U,\Lambda,d)\|\nabla(u-v)\|_{\underline{L}^2(U_r)}.
\end{aligned}$$

Finally, we calculate the H^2 norm of $\bar{u}$. Because it is the solution of $-\nabla\cdot\bar{\mathbf{a}}\nabla\bar{u} = \lambda^2 u_0$, we apply the classical H^2 estimate of elliptic equation (see [ES98])

$$\begin{aligned}
\Lambda^{-1}\|\bar{u}\|_{\underline{H}^2(U_r)} &\leqslant \lambda^2\|u_0\|_{L^2(U_r)} \\
(\text{ Using } eq.\ (4.4.1)) &\leqslant \lambda\Lambda\|\nabla(v-u)\|_{L^2(U_r)} \\
\Longrightarrow \|\bar{u}\|_{\underline{H}^2(U_r)} &\leqslant \lambda\Lambda^2\|\nabla(v-u)\|_{\underline{L}^2(U_r)}.
\end{aligned}$$

□

Remark. If we use Poincaré's inequality to $\Delta\bar{u}$ to get the $\underline{L}^2(U_r)$ of $\nabla\bar{u}$, we will get a factor of r, which is less optimal, but the regularization reduces this factor to λ^{-1}.

4.4.2 Proof of main theorem

With all these tools in hand, we can now prove theorem 4.1.1. We will denote by $\mathcal{R}(\lambda,\mu,r,a,d,U)$ the right hand side of eq. (4.3.3), so we aim to prove that

$$\|v-w\|_{\underline{H}^1(U_r)} \leqslant \mathcal{R}(\lambda,\mu,r,\mathbf{a},d,U,\bar{v}).$$

Proof. We take the first and second equations in the eq. (4.1.3) and use the equation eq. (4.1.1)

$$\begin{aligned}
-\nabla\cdot\bar{\mathbf{a}}\nabla\bar{u} &= \lambda^2 u_0 \\
&= f + \nabla\cdot\mathbf{a}\nabla(v+u_0) \\
&= -\nabla\cdot\mathbf{a}\nabla(u-v-u_0).
\end{aligned}$$

This is in the frame of theorem 4.3.1 thanks to the classical H^2 theory that $\bar{u} \in H^2(U_r)$. We apply theorem 4.3.1 with abuse of notation of the two scale expansion

$$w := \bar{u} + \sum_{k=1}^{d} \partial_{x_k}(\bar{u} \star \zeta)\phi_{e_k}^{(\lambda)}.$$

Then we obtain

$$\|w - (u - v - u_0)\|_{\underline{H}^1(U_r)} \leqslant \mathcal{R}(\lambda, 0, r, \mathbf{a}, d, U, \bar{u}). \tag{4.4.2}$$

The third equation of eq. (4.1.3) $(\lambda^2 - \nabla \cdot \mathbf{a}\nabla)\widetilde{u} = (\lambda^2 - \nabla \cdot \bar{\mathbf{a}}\nabla)\bar{u}$ is also of the form of the theorem theorem 4.3.1, so we obtain

$$\|\widetilde{u} - w\|_{\underline{H}^1(U_r)} \leqslant \mathcal{R}(\lambda, \mu, r, \mathbf{a}, d, U, \bar{u}). \tag{4.4.3}$$

We combine this two estimates and use the triangle inequality to obtain

$$\|(v + u_0 + \widetilde{u}) - u\|_{\underline{H}^1(U_r)} \leqslant \mathcal{R}(\lambda, 0, r, \mathbf{a}, d, U, \bar{u}) + \mathcal{R}(\lambda, \lambda, r, \mathbf{a}, d, U, \bar{u}). \tag{4.4.4}$$

It remains to see how to adapt $\mathcal{R}(\lambda, 0, r, \mathbf{a}, d, U, \bar{u})$ in a proper way in the context of eq. (4.1.1). We plug in the formula in lemma 4.4.1 to separate all the norms of H^1 and H^2 and use $\mu < \lambda$.

$$\begin{aligned}\mathcal{R}(\lambda, \mu, r, \mathbf{a}, d, U, \bar{u}) \leqslant C(U, \Lambda, d) \Big[\lambda + \lambda \mathcal{X}_1 + \left(1 + l(\lambda)^{\frac{1}{2}} \lambda^{\frac{1}{2}}\right) \mathcal{X}_2 \\ + \left(l(\lambda)^{\frac{1}{2}} \lambda^{\frac{1}{2}} + 1\right) \left(\lambda + \frac{1}{r} + \frac{1}{l(\lambda)}\right) \mathcal{Y}_1 \Big] \|v - u\|_{\underline{H}^1(U_r)}.\end{aligned}$$

We check the fig. 4.2 and notice that the largest term is $l(\lambda)^{-\frac{1}{2}} \lambda^{\frac{1}{2}} \mathcal{Y}_1$, so we obtain that the factor is of type $\mathcal{O}_s\left(C(U, \Lambda, s, d)(\log r)^{\frac{1}{s}} l(\lambda)^{\frac{1}{2}} \lambda^{\frac{1}{2}})\right)$ as desired. □

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